Wykład 2, *11.10.2010*

1. **Notation - pure strategies** $N = 1, 2, ..., n$, $n = |N|$ - set of players.
   $A_i \neq \emptyset$, $i = 1, \ldots, n$ - set of actions (pure strategies) of player $\tilde{i}$. $A := X_{i=1}^{n} A_i$.
   $u_i : A \to \mathbb{R}$ - the payoff of player $\tilde{i}$, $i = 1, \ldots, n$ $(u_i(a_1, \ldots, a_n))$.
   $(a_1, \ldots, a_n)$ - the profile. $u_i(a)$ - the payoff of player $\tilde{i}$ from the profile $a \in A$.
   Sometimes $(a_1, \ldots, a_i, \ldots, a_n) \equiv (a_i, a_{-i})$.

2. **Notation - mixed strategies**
   $\sum_i := \{\sigma_i : A_i \to [0,1] : \sum_{k=1}^{m_i} \sigma_{ik} = 1, \; \sigma_{i_k} \geqslant 0\}$ - the set of all mixed strategies of player $\tilde{i}$.
   $\sigma = (\sigma_j)_{j \in N} = (\sigma_1, \ldots, \sigma_n)$ - the profile of the game. $\sum := X_{i=1}^{n} \sum_i$ - the set of all profiles. σ_{-i} - subprofile without player $\tilde{i}$.
   $u_i(\sigma) = u_i(\sigma_i, \sigma_{-i})$ - payoff of player $\tilde{i}$ from the profile σ.
   The game is finite if $m_i = |A_i| < \infty$, $i = 1, \ldots, n$. $\sigma_{ih} \equiv x_{ih}$.

3. **GS**:=<N, $(A_i)_{i \in N}$, $(u_i)_{i \in N}$ >.

4. **Nash Equilibrium in pure strategies** of GS is the profile (of pure strategies) $a^* = (a_1^*, \ldots, a_n^*) \in A$ such that $\forall_{i \in N} \; \exists_{a_i \in A_i} \; u_i(a_i^*, a_{-i}^*) \geqslant u_i(a_i, a_{-i}^*)$. No player has an incentive to change individually her pure strategy.

5. **Mixed strategy** σ_i of $\tilde{i}$ player is a probability distribution over A_i.
   $\sigma_i(\sigma_{i_1}, \ldots, \sigma_{m_i})$ where $m_i = |A_i|$, $\sigma_i \geqslant 0$.
   E.g. mixed strategy $\sigma_i = x_i = (x_{i1}, \ldots, x_{im_i}) \in \mathbb{R}^{m_i} \; \forall_{i \in N} \; \sum_{h=1}^{m_i} x_{ih} = 1$.

6. **Def.** $\Delta_i := \{x_i = (x_{i1}, \ldots, x_{im_i}) \in \mathbb{R}^{m_i} \in \mathbb{R}^{m_i}\text{:} \; \sum_{h=1}^{m_i} x_{ih} = 1, \; x_{ih} \geqslant 0 \; \forall_{h \in A_i}\}$ is called **unit symplex** of the player $\tilde{i}$.

7. **Def.** The payoff of player $\tilde{i}$ from the profile $x = (x_1, \ldots x_n)$ (of mixed strategies) is the expectation value of u_i : $\tilde{u}_i(x) := \sum_{a \in A} u_i(a) x(a) = u_i(x)$ - average payoff.

8. **Lemma - linearity with repsect to coordinates**
   $\forall_{i \in N} \; \forall_{j \in N} \; u_i(x_1, \ldots, \sum_{k=1}^{m_j} x_{jk} e_j^k, \ldots, x_n) = \sum_{k=1}^{m_j} x_{jk} u_i(x_1, \ldots, e_j^k, \ldots, x_n)$.

9. **Def.** The profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ of the strategic game GS=<N, $(A_i)_{i=1}^n$, $(u_i)_{i=1}^n$ > is **Nash Equilibrium** $\Leftrightarrow \forall_{i \in N} \forall_{\sigma_i \in \sum_i} \; u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant (\sigma_i, \sigma_{-i}^*)$. No player has an incentive to change individually her strategy profile.

Wykład 3, *18.10.2010*

10. **Def. Support** of $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{im_i})$ of player $\tilde{i}$: $supp\sigma_i := \{a_k \in A_i : \sigma_{ik} > 0\}$.

11. **Theorem - payoffs from pure strategies** $x_i = \sum_{k=1}^{m_i} e_i^k x_{ik}$, $i = 1, \ldots, n$. Fix player $\tilde{i}$. Let $e_i^{k_1}$, $e_i^{k_2} \in suppx_i$ (i.e. $p_1 := x_{ik_1} > 0$, $p_2 := x_{ik_2} > 0$). Then $x = (x_1, \ldots, x_n)$ is NE $\Rightarrow u_i(e_i^{k_1}, x_{-i}) = u_i(e_i^{k_2}, x_{-i})$ $\forall_{i \in N}$.

12. **Consequence** Let $x^* = (x_1^*, \ldots, x_n^*)$ be a NE profile. Then $\forall_{i \in N}$ $\forall_{e_i^k \in suppx_i^*}$ $u_i(x_i^*, x_{-i}^*) = u_i(e_i^k, x_{-i}^*)$.

13. **Theorem** x^* is NE $\Leftrightarrow \forall_{i \in N}$:

    (i) $u_i(s', x_{-i}^*) = u_i(s'', x_{-i}^*)$, where s', $s'' \in suppx_i^*$

    (ii) $u_i(s', x_{-i}^*) \leqslant u_i(s'', x_{-i}^*)$, where $s' \notin suppx_i^*$, $s'' \in suppx_i^*$

14. **Lemma** x^* is NE $\Leftrightarrow \forall_{i \in N}$ $\forall_{e_i^k \in A_i}$ $u_i(e_i^k, x_{-i}^*) \leqslant u_i(x_i^*, x_{-i}^*)$.

15. **Def.** Strategy $\sigma_i \in \Sigma_i$ **strictly dominates** $\eta_i \in \Sigma_i \Leftrightarrow \forall \sigma_{-i} \in \Sigma_{-i}\ u_i(\sigma_i, \sigma_{-i}) > u_i(\eta, \sigma_{-i})$.
16. **Def.** Strategy $\sigma_i \in \Sigma_i$ **weakly dominates** $\eta_i \in \Sigma_i \Leftrightarrow \forall \sigma_{-i} \in \Sigma_{-i}\ u_i(\sigma_i, \sigma_{-i}) \geqslant u_i(\eta, \sigma_{-i})$ and $\exists \sigma_{-i} \in \Sigma_{-i}$ for which $' >'$.
17. **Def.** $\sigma_i \in \Sigma_i$ **dominates** $\eta_i \in \Sigma_i \Leftrightarrow \forall i\ \sigma_{-i} \in \Sigma_{-i}\ u_i(\sigma_i, \sigma_{-i}) \geqslant u_i(\eta_i, \sigma_{-i})$.
18. **Corollary** (Mixed) strategy which dominates each pure strategies of a player dominates each her strategy.
19. **Def.** Let $X, Y \neq \emptyset$. γ is a **correspondence** from X to Y iff. $\forall x \in X\ \gamma(x)$ is a (well defined) subset of Y, $\gamma : X \Rightarrow Y$, $\gamma : \rightarrow 2^Y$ (correspondence is a multivalue function).
20. **Brouwer Theorem** Let C - nonempty, compact, convex subset of $\mathbb{R}^m$. Let $f : C \rightarrow C$ - a continuous function. Then there exists a point: $\exists c \in C$ such that $f(c) = c$.
21. Correspondence $\psi : K \Rightarrow K$ (K doesn't contain $\emptyset$) has a fixed point $x \in K$ $\Leftrightarrow x \in \psi(x)$.
22. **Def.** $\gamma : E \rightarrow F$, $E, F \subset \mathbb{R}^m$. We define the graph of correspondence: $Gr\gamma = \{(x, y) \in E \times F : y \in \gamma(x)\}$. γ is **closed at** $x \in E \Leftrightarrow x^n \rightarrow x$, $y^n \rightarrow y$: $y^n \in \gamma(x^n) \Rightarrow y \in \gamma(x)$. γ is **closed** if γ is closed at all $x \in E$.
23. **Kakutani Theorem** $\mathbb{R}^n \supset X \neq \emptyset$ - compact, convex. $f : X \Rightarrow X$: 1) $\forall x \in X\ f(x)$ is nonempty, convex; 2) Grf is closed . Then f has a fixed point $\exists x \in X : x \in f(x)$.

Wykład 5, *08.11.2010*

24. **Best reply correspondence** $\forall i \in N\ \forall \sigma_{-i} \in \sum_{-i}\ B_i(\sigma_{-i}) = \{\sigma_i \in \sum_i : u_i(\sigma_i, \sigma_{-i}) \geqslant u_i(\tilde{\sigma}_i, \sigma_{-i})\ \forall \tilde{\sigma}_i \in \sum_i\}$ - the set of replies (of $\tilde{i}$ to σ_{-i})
$B_i : \sum_i \to 2^{\sum_i}$, $i = 1, \ldots, N$: **best reply correspondence** (of $\tilde{i}$)
$B : \sum \to \prod_{i=1}^n 2^{\sum_i}$, $B(\sigma) := \prod_{i=1}^n B_i(\sigma_{-i})$ - **best reply correspondence of GS**

25. **NE of GS** is a profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ such that $\forall i \in N\ \sigma_i^* \in B_i(\sigma_{-i}^*)$.
Observation: both definitions of NE are equivalent (proof by inspection).
$\equiv \sigma_* \in B(\sigma^*) \to \sigma^*$ is NE, when such thing happens.

26. **Theorem, J.E. Nash, 1950, Nobel prize - 1994** Every finite strategy game (GS) has a Nash equilibrium.

27. **Continuous set of strategies, Theorem (Glicksberg)** - GS $\forall i \in N$, $A_i \in \mathbb{R}^{m_i}$ is a nonempty and compact subset of $\mathbb{R}^{m_i}$, $\forall i \in N$, $u_i : A \to \mathbb{R}$ is continuous. Then GS has a NE.

28. **Extensive games** - decision are taken sequensively.

Wykład 6, *15.11.2010*

29. Game Tree:
    (a) **nodes** - root (initial mode), decision nodes (together with root), terminal nodes (sometimes: term used for last nodes, when no decision is taken)
    (b) **branches**
    (c) **information sets** - collection of nodes (singletons usually)
    (d) **player labels** - determine the decisions of the players
    (e) **action labels** - each branch has to be labelled
    (f) **payoffs** - defined on some subsets of the tree.

30. **Def. EG (Extensive Games, with Complete/Perfect Information:** $< I, H, P, (\succeq_i)_{i \in I} >$, where:
    - I - set of players
    - H - set of sequences, set of histories:
      - if $(a^k)_{k=1}^K \in H$ ($K \leqslant \infty$) and $L < K$ then (a $(a^k)_{k=1}^L \in H$)
      - if $(a^k)_{k=1}^\infty$: $(a^k)_{k=1}^L \in H$ $\forall L > 0$, then $(a^k)_{k=1}^\infty \in H$
      - $\emptyset \in H$ - empty history
      - $h \in H$ - histories
      - all elements of all $h \in H$ belong to **A** - set of all actions of all players
    - P - player function: $P : H \backslash Z \to I$, $h \in H \backslash Z$, $P(h)$ - the label of player, who moves after h
    - $\succeq_i$ - relation of preferences of i over Z.

31. **Def.** $h = (a^k)_{k=1}^K$ is **terminal** if h is infinite sequence or there is no action a^{k+1} such that $(a^k)_{k=1}^{K+1} \in H$. In the second case we say that history is **terminated**.
    Z - the set of all terminal histories.

32. **Def.** EG is **finite** if H is finite. EG has a **finite horizon** if the longest history is finite.

33. **Def. EG**$=< I, H, P, (u_i)_{i \in I} >$.

34. **Def. A(h)**$= \{a \in A : (h, a) \in H\}$ - the set of all actions of the player $P(h)$ after h (only in the one moment after).

35. **Def. Strategy**: $i \in I$, $A_i := \{a \in A : \exists h \in H \backslash Z : P(h) = i \wedge (h, a) \in H\}$ - all actions of player $\tilde{i}$.
    For $h \in H \backslash Z$, $P(h) = i$: $A_i(h) := \{a \in A_i : (h, a) \in H\}$ - all actions of $\tilde{i}$ after h.
    Strategy (of $\tilde{i}$) - a function $s_i : \{h : P(h) = i\} \to A_i : s_i(h) \in A_i(h)$.

Wykład 7, *22.11.2010*

36. **Def. Profile (of strategies) in EG**: $s := (s_1, \ldots, s_n)$, where s_i - strategy of i.

37. **Def. Outcome of the profile s in EG**, $h \in Z$ is constructed in the following way:
$P(\emptyset)$ applies $s_{P(\emptyset)}(\emptyset)$ playing the action $a^1 := s_{P(\emptyset)}(\emptyset)$. If $(a^1) \in Z$ then it is denoted $o_h(s)$ and called outcome of s. If $a^1 \in H \backslash Z$ the player $P((a^1))$ uses her strategy $s_{P((a^1))}((a^1))$ and applies the action a^2: $a^2 := s_{P((a^1))}((a^1)) \in A((a^1))$. If history $(a^1, a^2) \in Z$ - then stop, we call it $o(s)$. Otherwise we continue.

38. **Def. o(s) of the profile s** is the history $h \in Z$ $o(s) \in Z$: $o(s) = (a^k)_{k=1}^K$, $K \leqslant \infty$, such that $a^1 = s_{P(\emptyset)}(\emptyset)$, $a^{k+1} = s_{P((a^1,a^2,\ldots,a^k))}((a^1, a^2, \ldots, a^k))$, $1 \leqslant k < K$.

39. **Strategic form of EG**, $EG =< N, H, P, (u_i)_{i \in I} >$ generates SG.
NF of EG (normal form representation of EG): $GS :< N, (s_i)_{i=1}^n, (\bar{u}_i)_{i=1}^n >$, where s_i - the set of all strategies of $\tilde{i}$ in EG, $\bar{u}_i$ - the payoff funtions of $\tilde{i}$, $\bar{u}_i(s) := u_i(o(s))$.

40. **NE in EG**$=< n, H, P, (u_i) >$ is the profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*) : \forall i \in N$ $\forall r_i \in s_i$ $u_i(o(s_i^*, s_{-i}^*)) \geqslant u_i(o(r_i, s_{-i}^*))$.

41. **Def.** $\forall h \in H \backslash Z$: **subgame GE(h) of GE**$=< N, H, P, u_i >$ is the following extensive game $GE(h) =< N, H'(h), P'_h(u'_i)_{i=1}^n >$, where:
$H'(h)$ - the set of all h': $(h, h') \in H$; H' has an additional element $\emptyset$
$P'_h : H'(h) \to N : P'_h(h') = P((h, h'))$, $P'(\emptyset) = P(h)$
$u'_i(h') = u_i(h, h')$. $EG(\emptyset) = EG$; all other subgames - proper subgroups.

42. **Def. SPE** in EG is $s^* = (s_1^*, \ldots, s_n^*)$: $\forall i \in N$ $\forall GE(h)$ of EG the restriction of s^* to $GE(h)$ is a NE in $GE(h)$.

Wykład 8, *29.11.2010*

43. A profile $s^* = (s_1^*, \dots, s_n^*)$ is **SPE** (subgame perfect equilibrium) if: $\forall i = 1, \dots, n$ $\forall_{h \in H \setminus Z}$: $P(h) = i$, $u_i(o_h(s_i^*, s_{-i}^*)) \geqslant u_i(o_h(s_i, s_{-i}^*))$.
44. **Method of backward induction**.
45. **Games with perfect information** - players' function is single - valued and each player knows all the previous actions of the player.
46. Games is **finite** if $|H| < \infty$ and it has finite horizon. For such games MBI gives unique SPE.

Wykład 9, *06.12.2010*

47. **Example** - Simple Poker (EG with Imperfect Information).
48. **Def.** GS jest **grą o sumie stałej**: $\exists c \in \mathbb{R}$: $\forall a \in A$ $\sum_{i=1}^n u_i(a) = c$.
    GS jest **grą o sumie zerowej** $\Leftrightarrow c = 0$.
49. $v_1 := \max_{\sigma_1 \in \Sigma_1} \min \sigma_2 \in \Sigma_2 u_1(\sigma_1, \sigma_2)$ - **maximin**.
    $v_2 := \min_{\sigma_2 \in \Sigma_2} \min \sigma_1 \in \Sigma_1 u_1(\sigma_1, \sigma_2)$ - **minimax**.
50. **Def.** (σ_1^*, σ_2^*) jest **punktem siodłowym GS0** (saddle point), gdy: $u_1(\sigma_1, \sigma_2^*) \leqslant u_1(\sigma_1^*, \sigma_2^* \leqslant u_1(\sigma_1^*, \sigma_2)$ $\forall \sigma_i \in \Sigma_i$.
51. **Wartość gry**:= $u_1(\sigma_1^*, \sigma_2^*)$.
52. **Spostrzeżenie (lemat)** Punkt siodłowy jest równowagą Nasha.
53. **Twierdzenie o minimaksie (von Neumanna, 1928)** Dla każdej GS0 (dwuosobowej gry strategicznej o sumie 0):
    (a) istnieje punkt siodłowy
    (b) $\exists!$ v^*: $v_1 = v_2 = v^*$
    (c) $\sigma_1^*, \sigma_2^*)$ jest punktem siodłowym $\Rightarrow u_1(\sigma_1^*, \sigma_2^*) = v^*$
    (d) $\sigma_1^*, \sigma_2^*)$ jest punktem siodłowym $\Leftrightarrow [\sigma_1^* \in arg \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2)$, $\sigma_2^* \in arg \min_{\sigma_2} \max_{\sigma_1} u_1(\sigma_1, \sigma_2)$.

Wykład 10, *13.12.2010*

54. EG is **EG with Perfect Information (EGwPI)** if P is single valued and each player, when choosing an action, knows all the actions and corresponding players in previous time steps.

55. **Theorem (Existence, Kuhn)** Every finite EGwPI has a pure SPE. If the players have unique preferences of the choice of their actions, then the SPE is unique.

56. **EG with Imperfect Information** e.g. poker.
EG with Simultaneous Moves - player function is not single valued (H is a sequence of vectors of actions).

57. **Remark** For any strategic game there exists EGwSM in which every terminal history has lenght 1, the set Z is the set of action profiles in SG, $P(\emptyset) = N$, and the set of actions $A_j(\emptyset)$ of player j is the set of j actions (strategies) in SG.

58. **Remark** Any SG can be represented as EGwII.
COALITIONAL GAMES (cooperation games)

59. **Assumptions**
    - existence of universal currency, which units are the same for all players and players can exchange this currency among themselves (transfer utility)
    - **superadditivity** - CG is superadditive $\Leftrightarrow$ $S, T \subset 2^n$, $S \cap T = \emptyset$ $v(S \cup T \geqslant v(S) + v(T)$
    - the players form the grand coalition

60. **Def.** CGwTU is a pair $< N, v >$, $|N| < \infty$, $v : 2^N \to \mathbb{R}$ - value function, $v(\emptyset) = 0$, $N = \{1, \ldots, n\}$.

61. **Coalition** $S \subset N$, $\emptyset$ - empty coalition, N - grand coalition, $v(S)$ - power value of S (payoff).

62. **Def.** $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ - payoff vector of CG.
    - x is an **allocation** (or **group rational**) if $\sum_{i=1}^n x_i = v(N)$.
    - x is **individually rational** if $x_i \geqslant v(\{i\})$, $i = 1, \ldots, n$
    - x is **coalitionally rational** if $\forall S$ $\sum_{j \in S} x_j \geqslant v(S)$.

63. **Def.** The payoff vector x is called **imputation** if it is group rational and individually rational.

64. **Def. Core** of $< N, v >$ is the set of coalitionally rational (stable) imputations $C := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N)\ \forall S \sum_{j \in S} x_j \geqslant v(S)\}$.

Wykład 11, *20.12.2010*

65. **Def.** The imputation is **stable** if it is coalitionally rational.

66. **Def. Shapley value** $\Phi(v)$ of CG $< N, V >$ is a vector $(\Phi_1(v), \dots, \Phi_n(v))$, $\Phi_i \in \mathbb{R}$, $i = 1, \dots, n$, which satisfy:
    (i) **efficiency:** $\sum_{i=1}^{N} \Phi_i(v) = v(N)$
    (ii) **Symmetry:** if $v(S \cup \{i\}) = v(S \cup \{j\})$ $\forall S$, $i, j \notin S$, then $\Phi_i(v) = \Phi_j(v)$
    (iii) **Dummy player:** if $v(S \cup \{i\}) = v(S)$ $\forall S$, $i \notin S$, then $\Phi_i(v) = 0$
    (iv) **Additivity:** if u, v - characteristic functions, then $\forall i$ $\Phi_i(u + v) = \Phi_i(u) + \Phi_i(v)$.

67. **Def. Shapley value of player** $\tilde{i}$ is the i-th coordinate of the Shapley value Φ $(\Phi_i(v))$.

68. **Theorem** There is a unique Shapley value of CG $< N, v >$
    $\Phi_i = \sum_{S, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!} \Delta_i(S)$, $i = 1, \dots, n$, $\Delta_i(S) = v(S) - v(S \cup \{i\})$.

WYKŁAD 12, *03.01.2011*

69. **Def. Simple game**, CG $< N, v >$: $\forall S \in 2^N$, $v(s) \in \{0,1\}$, examples: the unionianity game, majority, the weighted voting game.

70. **Def.** If for $i \in S$ $v(S \backslash \{i\}) = 0$ $v(S) = 1$, then i is **critical player**.

71. **Fact** For simple games: $\phi_i(v) = \frac{1}{n!} \sum_{S\text{: i is critical}} (|S|-1)!(n-|S|)!$, $i = 1, \ldots, n$
    ITERATED GAMES (repeated)

72. Let's consider $GS : < N, A_i, u_i >$ (one shot, only pure strategies). We assume that at time $t = 1, 2, \ldots$, the players know all the actions of all players in the previous rounds.
    $a^t = (a_1^t, a_2^t, \ldots, a_n^t)$, $n = |N|$
    $h^t = (a^0, a^1, \ldots, a^{t-1})$ - history at time t, a^0 - empty profile
    History is **terminal** $\Leftrightarrow$ infinite. Formally: $(a^0, a^1, a^2, \ldots)$.

73. **Examples of strategies:** All C; All D; Tit for Tat; Win-Stay, Lose-Shift (Pavlov strategy); Brutal

74. **Def. A strategy** of i is the infinite sequence of functions, $s_i = (s_i^2, s_i^2, \ldots)$, $s_i^t : H^t \to A_i$, $t = 1, 2 \ldots$, $s_i^t(h^t)$ describes an action of i after history h^t at time t.

75. **Example (Grimm Trigger)**: t=1 C, and plays C till the opponent plays D for the first time and then D all the time, $(s_1, \ldots, s_n)$.
    $(s_1, \ldots, s_n)$ - profile; strategies - infinite sequences (each s determines a terminal history)

76. **Def. The (discounted) payoff** of i from $h = (a^1, a^2, \ldots)$:
    $u_i(h) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t)$.

77. **Observation** $s = (s_1, \ldots, s_n)$ determines a terminal history h, $u_i(h) = u_i(s)$.

78. **Def. Profile** s **is NE** in IRG (infinitely repeated games) if $\forall i = 1, \ldots, n$ $u_i(s_i, s_{-i}) \geqslant u_i(\tilde{s}_i, s_{-i})$ $\forall \tilde{s}_i$

79. **Example** Infinite Prisoner's Dilemma (IPD) - all D is NE. In 2-persons IPD $s := (GT, GT)$ is NE provided $\delta \geqslant \frac{1}{2}$.

Wykład 13, *10.01.2011*

80. **Additional knowledge** - iterated games: Axelrod's Tournament (1979), Ecological Tournament - Tit for Tat is the winner strategy in both cases.
Properties of TFT (and some other strategies): niceness (never defect first), provocability (retaliate immediately after being cheated), forgiveness (retaliate once and forgive).
Folk Theorem Almost every payoff can be realized in a NE of IG.

81. **Evolutionary Game Theory (ETG)**, John Maynard Smith
EGT is one of the most important frameworks for studying evolution in different scientific domains like biology, social sciences, economics.
EGT describes behaviour of large populations of individuals who wplay IG.

82. **Notation** - players (A,B), birth rates, frequency of players (N_1/N, N_2/N), number of new players born from t to $t + \Delta t \sim$ is proportional to number of players in t and $\Delta(t)$ (small).

83. **Statement** $f_A(t+\Delta t) > f_A(t) \Leftrightarrow a > b$ - frequency of A increases if $a > b$.

84. **Evolutionary scenario**

    a) large population of identical players, each has a fixed strategy

    b) players are matched pairwise and play SG; this game is symmetric (as players are identical)

    c) players produce offsprings - number of them (of a player) is proportional to his (her) payoff; offspring heritates parent strategy, parents do not die

    d) come back to point a) in next time step.

85. **Another description**: Evolutionary Game (EG) is a strategic game played in populations of individuals according to the evolutionary scenario.

86. **Example (Hawk-Dove)** - replicator (equation, dynamics).

87. **Evolutionary Scenario - notation**:

    - GS: $< \{1,2\}, A_i, u_i >$, $i = 1, \ldots, n$
    - $N_i(t)$ - mass (# of) $\tilde{i}$ players at t, $N = \sum_{i=1}^n N_i$, players are identical
    - $x_i(t) = x_i = \frac{N_i}{N}$, $x = (x_1, \ldots, x_n)$, $x(t) = \sum_{k=1}^n e^k x_k$
    - $x_i = x_i(t) = \frac{N_i(t)}{N(t)}$ - frequency of $\tilde{i}$
    - $u(e^i, x)$ - payoff of strategy $\tilde{i}$ (against x), when the state of system (population) is $x = x(t) = (x_1, \ldots, x_n)$
    - $u(x, x) = \sum_{i=1}^n x_i u(e^i, x)$ - mean (average) payoff of a player (in the population)

- assumption: $\dot{p}_i = p_i u(e^i, x)$, $i = 1, \ldots, n$
  $\frac{\dot{p}}{p}$ - rate of production of $\tilde{i}$ equals the mean payoff of $\tilde{i}$ strategy
  replicator equation (dynamics):
  $\dot{x}_i(t) = x(t)(u(e^i, x) - u(x, x))$, $i = 1, \ldots, n$.
- fitness=payoff.

Wykład 14, *17.01.2011*

88. **Basic evolutionary scenario, c.d.**:

 - a_i - birth rate of $\tilde{i}$, $a_i \sim u(e^i, x)$, $(a_i = ku(e^i, x))$; we assume that $a_i = u(e^i, x)$, $k = 1$.
 - $N_A(t + \Delta t - N_A(t) = aN_A(t)\Delta t$; if we have $i = 1, \dots, n$, we get $N_i(t + \Delta t) - N_i(t) = a_i N_i(t)\Delta t$; finally we get $\dot{N_i} = N_i a_i$
 - $\frac{\dot{N_i}}{N_i} = u(e^i, x) = a_i$, we get $\dot{x}_i = x_i(u(e^i, x) - u(x, x))$, $i = 1, \dots, n$ – **replicator dynamics equation (RDE)** – system of $n - 1$ equation as we know that $x_1 + \dots x_n = 1$.
 - $\frac{\dot{x}_i}{x_i} = u(e^i, x) - u(x, x)$ - speed of $\tilde{i}$ frequency change.

89. **Example** Assume symmetric 2-person game A:
$u(e^i, x) = (Ax^T)_i$, $u(x, x) = xAx^T$, $\dot{x}_i = x_i((Ax)_i - xAx^t)$, $i = 1, \dots, n$, $x = (x_1, x_2)$, $n = 2$, $x_2 = 1 - x_1$, $\dot{x}_1 = x_1(1 - x_1)[(Ax)_1 - (Ax)_2]$.

90. **Hawk-Dove Example**

91. **Other statements:**

 - $u(e^i, x) > u(e^j, x) \Rightarrow \frac{d}{dx}(\frac{x^i}{x^j}) > 0$
 - unit simplex $(x_1 + \dots + x_n = 1)$ is invariant in RD
 - $x_i = 0$ at $t_0 \Rightarrow x_i(t) = 0$, $t \leqslant t_0$

92. **Definition** In symmetric 2-person games strategy $\hat{x}$ of a player is **Nash strategy** if profile $(\hat{x}, \hat{x})$ is NE.

93. **Theorem** In 2-person symmetric games: if $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ is a Nash strategy, then $\hat{x}$ is the critical point of RD.

94. **Other theorems:**

 - Liapunov stable critical points are Nash strategies.
 - Nash strategies which are Evolutionary Stable Strategies are (locally) asymptotically stable.
 - Frequency of strongly dominated strategy decreases with $t \to \infty$.