1. Laplace eq. (a) & Poisson eq. (b)
   $\Omega \subset \mathbb{R}^n$ – open, elliptic equations:
   (a) $\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) = 0$
   (b) $\Delta u = f$
2. Harmonic function in Ω
   u – if $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω.
3. Exercises
   (i) The Laplace operator Δ is invariant under rotations A – a matrix of rotation in $\mathbb{R}^n$ i.e. $\Delta(Au) = \Delta u$.
   (ii) Assume $\Omega = \mathbb{R}^n$. If u is a radial solution $\Delta u = 0$ i.e. $u(x) = v(r)$, $r = |x|$ then v satisfies the equation: $v'' + \frac{n-1}{r} v' = 0$. Then the radial solutions are $u(x) = const$

$$v(r) = \begin{cases} b \log r + c & n = 2 \\ \frac{b}{r^{n-2}} & n \geqslant 3 \end{cases}$$

4. Fundamental solution of the Laplace eq.
   The function

$$E(x) = \begin{cases} \frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(2-n)\omega_n} \frac{1}{|x|^{n-2}} & n \geqslant 3 \end{cases}$$

   $\omega_n = |B^n(0,1)|$
   $\Delta E(x) = 0$ in $\mathbb{R}^n \backslash \{0\}$
   $E \in L^1_{loc}(\mathbb{R}^n)$
5. Theorem
   Assume $f \in C^2_o(\mathbb{R}^n)$ – twice continuously differentiable with compact support in $\mathbb{R}^n$. Define $u(x) = E * f(x) = \int_{\mathbb{R}^n} E(y) f(x-y) dy$. Then
   (i) $u \in C^2(\mathbb{R}^n)$
   (ii) $\Delta u = f$ in $\mathbb{R}^n$ – u is the solution to the Poisson equation.
6. Divergence theorem
   $\Omega \subset \mathbb{R}^n$ – open and bounded, $\partial\Omega \in C^1$,
   $\bar{\omega} = (\omega_1, \dots, \omega_n) : \Omega \to \mathbb{R}^n$, $\omega \in C^1(\bar{\Omega})$
   $\int_\Omega div \bar{\omega} dx = \int_{\partial\Omega} \bar{\omega} \bar{n} d\sigma$.
7. Remark
   Assume $u \in C^2(\Omega)$ and harmonic and let $\bar{G} \subset \Omega$ and $\partial G \in C^1$ (smooth). Then
   $0 = \int_G \Delta u dx = \int_{\partial G} \frac{\partial u}{\partial \bar{n}} d\sigma$.
8. Theorem (Mean value formula)
   Assume $u \in C^2(\Omega)$ and u is harmonic. Then for every point $x \in \Omega$ and r: $0 < r < dist(x, \partial\Omega)$ we have
   $u(x) = \oint_{\partial B(x,r)} u(y) d\sigma(y) = \oint_{B(x,r)} u(y) dy$.
9. Theorem
   If $u \in C^2(\Omega)$ and for every point $x \in \Omega$ $u(x) = \oint_{\partial B(x,r)} u(y) d\sigma(y)$ then u is harmonic in Ω for every ball $\overline{B(x,r)} \subset \Omega$.
10. Theorem (Maximum principle)
    Assume Ω is open and bounded. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω. Then
    (i) $\max_{\bar{\Omega}} u(x) = \max_{\partial\Omega} u(x)$
    (ii) if Ω is connected and there exists $x_0 \in \Omega$ $u(x_0) = \max_{\bar{\Omega}} u(x)$ then $u = const$ (strong maximum principle).
11. Remark
    If Ω is unbounded, then the maximum principle does not need to be true.
12. Corollary
    $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $g \in C(\partial\Omega)$, Ω is open, bounded and connected

$$(*) \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

    Then, if $g > 0$ somewhere on $\partial\Omega$ then $u > 0$ everywhere on Ω.
13. Theorem (uniqueness)
    Ω – open and bounded, $f \in C(\Omega)$, $g \in C(\partial\Omega)$, Poisson equation:

$$(**) \begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

    There exists at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the problem.
14. Theorem
    Suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is solution to (**). Then for every $y \in \Omega$ we have
    $u(y) = \int_\Omega E(x-y) \Delta u(x) dx + \int_{\partial\Omega} (u(x)) \frac{\partial E}{\partial \bar{n}}(x - y) - E(x-y) \frac{\partial u}{\partial \bar{n}}(x)) d\sigma(x)$.
15. Green's function for the region Ω
    $G(x,y) = E(x-y) + h_y(x)$, where h_y is a harmonic function $\Delta h_y = 0$ in Ω, $h_y \in C^1(\bar{\Omega})$, $E(x-y) + h_y(x) = 0$ on $\partial\Omega$.
16. Theorem (Green's function representation of the solutions)
    If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution to (**), $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\partial\Omega)$ and G is Green's function for Ω then $u(y) = \int_\Omega G(x,y) f(x) dx + \int_{\partial\Omega} g(x) \frac{\partial G}{\partial \bar{n}}(x,y) d\sigma(x)$.
17. Inversion
    $x \in \mathbb{R}^n \backslash \{0\}$, $x^*_R = \frac{R^2 x}{|x|^2}$ is called a dual to x with respect to $\partial B(0,R)$. $x \mapsto x^*_R$ is inversion with respect to $\partial B(0,R)$.
    $|x^*| = \frac{R^2}{|x|}$.
18. Green's function for a ball
    $G(x,y) = E(x-y) - E(|y||x-y^*|)$
19. Theorem
    If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solves the problem

$$\begin{cases} \Delta u = 0 & \text{in } \partial B(0,R) \\ u = g & \text{on } \partial B(0,R) \end{cases}$$

    where $g \in C(\partial\Omega)$, then
    $u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{g(x)}{|x-y|^n} d\sigma$.

20. Poisson kernel
The function $K_R(x,y) = \frac{R^2-|y|^2}{n\omega_n R|x-y|^n}$ for $x \in \partial B(0,R)$, $y \in B(0,R)$.

21. Theorem (Poisson's formula for the ball)
Assume $g \in C(\partial B(0,R))$ and $u(y) = \int_{\partial B(0,R)} K_R(x,y)g(x)d\sigma(x)$. Then:

 (i) $u \in C^\infty(B(0,R))$
 (ii) $\Delta u = 0$ in $B(0,R)$
 (iii) $\lim_{y\to y_0} u(y) = g(y_0)$, $y_0 \in \partial B(0,R)$.

22. Exercises

 (i) Give another proof with the help of the formula $u(y) = \frac{R^2-|y|^2}{n\omega_n R}\int_{\partial B}\frac{g(x)}{|x-y|^n}d\sigma(x)$ that if u is continuous and $u(y) = \oint_{B(y,R)} u(x)dx = \int_{\partial B(y,r)} u(x)d\sigma(x)$ then $\Delta u = 0$ in Ω.
 (ii) If u is harmonic in Ω then u is analytic in Ω $\sum_\alpha \frac{D^\alpha u(y+0)}{\alpha!}(y-y_0)^n = u(y)$ (if $\alpha = (\alpha_1,\ldots,\alpha_n)$, then $\alpha! = \alpha_1!\ldots\alpha_n!$).
 (iii) $\Delta u = 0$ in $B(0,1)$, u is bounded on $B(0,1)$. Show that $\sup_{x\in B}(1-|x|)|\nabla u(x)| < C < \infty$.

23. Theorem
Let u be harmonic in Ω. Then for every ball $\bar{B} = \overline{B(x_0,R)} \subset \Omega$, for every α, $|\alpha| = k$ we have $D^\alpha u(x_0)| \leqslant \frac{C(n,k)}{R^{n+k}}||u||_{L^1(B(x_0,R))}$.

24. Theorem (Liouville's)
If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic ($\forall_{x\in\mathbb{R}^n}\Delta u(x) = 0$) and u is bounded, then $u = const$.

25. Theorem
$n \geqslant 3$, if u is solution to $\Delta u = f$ in $\mathbb{R}^n$ and u is bounded, then $u = \int_{\mathbb{R}^n} E(x-y)f(y)dy + c$.

26. Theorem
If (u_k) is a sequence of harmonic functions in Ω $u_k \to u$ then u is harmonic in Ω.

27. Theorem
u is harmonic in $\Omega \subset \mathbb{R}^n$ – open, bounded, $K \subset L \subset \Omega$ – compact subsets of Ω, then $\sup_K |D^m u| \leqslant \frac{C(n,m)}{dist(K,L)^m}\sup_L |u|$.

28. Remark
If $||u||_{L^\infty(\Omega)} < c$, then we have $\sup_K |D^m u| \leqslant \frac{C(n,n)}{dist(K,\partial\Omega)^m}\sup_\Omega |u|$.

29. Theorem (Arzela-Ascoli)
We have $f_n : K \subset \mathbb{R}^n \to \mathbb{R}$ (K – compact). Moreover:

 (i) f_n are uniformly bounded: $||f_n||_{L^\infty(K)} \leqslant C$
 (ii) f_n are equicontinuous on K i. e. $\forall_\varepsilon \exists_\delta \forall_{x,y\in K}\forall_{n\in\mathbb{N}}$ $||x-y|| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Then there exists a subsequence f_{n_k} which converges almost uniformly.

30. Theorem
Let u_n be a sequence of harmonic functions $u_n : \Omega \to \mathbb{R}$, which is uniformly bounded.
Then there exists a subsequence u_{n_k} such that it converges almost uniformly (uniformly on every $K \subset \Omega$, K – compact) and the limit function is harmonic ($\Delta u = 0$ in Ω).

31. Exercise (Harnack inequality)
If we have u – harmonic in Ω, $u \geqslant 0$. $\Omega' \subset\subset \Omega$ – open subset of Ω (compactly contained in Ω), then there exists $C = C(n,\Omega,\Omega')$ such that $\sup_\Omega u \leqslant C\inf_{\Omega'} u$.

32. Theorem
u_n are harmonic, $u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n \leqslant \ldots$ (we have a monotonic sequence of harmonic functions).
Assume that there exists a point $y_0 \in \Omega$ such that $\lim_{n\to\infty} u_n(y_0) = u_0$.
Then u_n converges almost uniformly and the limit function is harmonic.

33. Dirichlet functional
Let $\Omega \subset \mathbb{R}^n$ – open, bounded subset of $\mathbb{R}^n$, with smooth boundary ($\partial\Omega \in C^1$)
$\psi \in C(\partial\Omega)$ – given function
$K_\psi = \{w : w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $w|_{\partial\Omega} = \psi\}$ – class of functions.
We introduce a functional $\int_\Omega |\nabla u(x)|^2 dx = E(u)(\geqslant 0)$ – Dirichlet's functional.

34. Theorem (Dirichlet's principle)
Assume $u \in K_\psi$. The following conditions are equivalent:

 (i) $\Delta u = 0$ in Ω
 (ii) $\int_\Omega \nabla u \nabla \phi dx = 0$ $\forall_{\phi\in C_0^2(\Omega)}$
 (iii) $E(u) \leqslant E(w)$ $\forall_{w\in K_\psi}$

35. Lemma (du Bois-Reymond/fundamental lemma of the calculus of variations)
$f \in L^1_{loc}(\Omega)$ and $\int_\Omega f(x)\phi(x)dx = 0$ $\forall_{\phi\in C_0^\infty(\Omega)}$.
Then $f = 0$ almost everywhere.

36. Słaba pochodna (weak derivative)
$g, f \in L^1_{loc}(\Omega)$. g jest słabą pochodną f (tzn. $g = D^\alpha f$) $\Rightarrow \forall_{\phi\in C_0^\infty(\Omega)}$:
$\int_\Omega f(x)D^\alpha\phi(x)dx = (-1)^{|\alpha|}\int_\Omega g(x)\phi(x)dx$.
Dla $|\alpha| = 1$: $\forall_{\phi\in C_0^\infty(\Omega)}$:
$\int_\Omega f(x)\frac{\partial\phi}{\partial x_i}(x)dx = -\int_\Omega g(x)\phi(x)dx$.

37. Uwaga
Jeśli $u \in C^1(\Omega)$ to słaba pochodna=klasyczna pochodna.

38. Uwaga
Słaba pochodna $D^\alpha f \in L^1_{loc}(\Omega)$ jest zdefiniowana z dokładnością do zbioru miary 0.

39. Lemat
Jeśli słaba pochodna istnieje, to (z dokładnością do zbioru miary 0) jest wyznaczona jednoznacznie.

40. Przykład

- $f(x) = |x|$, $x \in [-1,1]$: słaba pochodna istnieje i jest równa $f'(x) = 2\chi_{[0,1]} - 1$.
- $H(x) = \chi_{[0,\infty)}$ nie posiada słabej pochodnej.

41. Przestrzeń Sobolewa $W^{1,p}(\Omega)$
Przestrzeń $\{u \in L^p(\Omega)$: słabe pochodne $\frac{\partial u}{\partial x_i}$ istnieją oraz $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$ dla $i = 1, 2, \dots, n\}$, $p \geqslant 1$.
Równoważne normy:
    - $||u||_{1,p,\Omega} = ||u||_{L^p(\Omega)} + \sum_{i=1}^n ||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)}$
    - $||u||_{1,p,\Omega} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}$
    - $||u||_{1,p,\Omega} = (\int_\Omega |u|^p dx + \sum_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^p dx)^{\frac{1}{p}}$

42. Twierdzenie
Przestrzeń $W^{1,p}(\Omega)$ wraz z normą $|| \cdot ||_{1,p,\Omega}$ jest przestrzenią Banacha.

43. Przestrzenie Sobolewa $W^{m,p}(\Omega)$
Przestrzenie Sobolewa w wyższych wymiarach, $|\alpha| \leqslant m$: $\{u \in L^p(\Omega)$: słabe pochodne $D^\alpha u$ do rzędu m włącznie istnieją oraz $D^\alpha u \in L^p(\Omega)$ dla $i = 1, 2, \dots, n\}$, $p \geqslant 1$, $m \in \mathbb{N}$.
Norma:
$||u||_{m,p,\Omega} = ||u||_{L^p(\Omega)} + \sum_{|\alpha|\leqslant m} ||D^\alpha u||_{L^p(\Omega)}$.

44. Przestrzeń Sobolewa $H^{1,p}(\Omega)$
Jest to uzupełnienie podprzestrzeni liniowej $\{u \in C^1(\Omega) : \int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx < \infty\} \subset C^1(\Omega)$ w normie $|| \cdot ||_{1,p,\Omega}$.

45. Przestrzenie Sobolewa $H^{m,p}(\Omega)$
$\{u \in C^m(\Omega) : \sum_{0\leqslant\|\alpha|\leqslant m} \int_\Omega |D^\alpha u| dx < \infty\} \subset C^m(\Omega)$ w normie $|| \cdot ||_{m,p,\Omega}$.

46. Uwaga
$H^{1,p}(\Omega) \subset W^{1,p}(\Omega)$.

47. $W_0^{1,p}$
Jest to domknięcie przestrzeni $C_0^\infty(\Omega)$ ($\subset W^{1,p}$) w normie $|| \cdot |||_{1,p,\Omega}$. $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$.

48. Uwaga

$$\tilde{u}(x) = \begin{cases} u(x) & \text{dla } x \in \Omega \\ 0 & \text{dla } x \in \mathbb{R}^n \backslash \Omega \end{cases}$$

Jeśli $u \in W^{1,p}(\Omega)$, to $\tilde{u}$ nie musi należeć do $W^{1,p}(\mathbb{R}^n)$ (chyba, że $u \in W_0^{1,p}(\Omega)$).

49. Twierdzenie
$C^\infty(\Omega)$ jest gęstą podprzestrzenią $W^{1,p}(\Omega)$.

50. Lemma
Assume $u \in W^{1,p}(\Omega)$ and $\eta \in C_0^\infty(\Omega)$ then $\eta u \in W^{1,p}(\Omega))$ and $D^\alpha(\eta u) = \eta D^\alpha u + u D^\alpha \eta$, $|\alpha| = 1$.

51. Mollifier $\Omega \subset \mathbb{R}^n$ – open, $\Omega_\varepsilon = \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$.

$$\eta(x) = \begin{cases} c \cdot \exp(\frac{1}{|x|^2-1}) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geqslant 1 \end{cases}$$

$\eta \in C_0^\infty(\Omega)$, $supp\eta = \overline{B(0,1)}$, $c > 0$ and such that $\int_{\mathbb{R}^n} \eta dx = 1$.
$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$, $\eta_\varepsilon \in C_0^\infty(\Omega)$, $supp\eta_\varepsilon = \overline{B(0,\varepsilon)}$, $\int_{\mathbb{R}^n} \eta_\varepsilon dx = 1$.

52. Standard mollifier
Function η if $f \in L^1_{loc}(\Omega)$.

53. Mollification of f
$f^\varepsilon = \eta_\varepsilon * f$ in Ω_ε, if $f \in L^1_{loc}(\Omega)$.
$f^\varepsilon = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y) dy$
$f^\varepsilon = \int_{\mathbb{R}^n} \eta_\varepsilon(y) f(x-y) dy = \int_{B(0\varepsilon)} \eta_\varepsilon(y) f(x-y) dy = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) f(y) dy$ ($x \in \Omega_\varepsilon$, fixed)

54. Properties of mollification
$f \in L^1_{loc}(\Omega)$:
    (a) $f^\varepsilon \in C^\infty(\Omega)$
    (b) $f^\varepsilon \to f$ a.e. in Ω_ε
    (c) if $f \in C(\Omega)$ then $f^\varepsilon \to^{\text{almost uniformly}} f$
    almost uniform convergence=uniform on every compactly conatined subset V on Ω (i.e. $V \subset\subset \Omega$)
    (d) if $f \in L^p(\Omega)$ then $f^\varepsilon \to f$ in $L^p_{loc}(\Omega)$,

55. Theorem (local approximation of Sobolev functions)
$\Omega \subset \mathbb{R}^n$ – open and connected.
Assume $u \in W^{1,p}(\Omega)$ for some $1 \leqslant p < \infty$. Define $u^\varepsilon = \eta_\varepsilon * u$. Then $u^\varepsilon \to u$ in $W^{1,p}_{loc}(\Omega)$ (convergence in $W^{1,p}(V)$ for every $V \subset\subset \Omega$).

56. Theorem (global approximation)
Ω – open, connected and bounded.
Assume $u \in W^{1,p}(\Omega)$ for some $1 \leqslant p < \infty$. Then there exists a sequence of function $u_m \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_m \to u$ in $W^{1,p}(\Omega)$.

57. Corollary
$W^{1,p}(\Omega) = H^{1,p}(\Omega)$ (Ω is bounded).
$W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$
But: if $\Omega = \mathbb{R}^n$ then $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.
$W_0^{1,p}(\Omega)$ – completion of $C_0^\infty(\Omega)$ in a Sobolev norm $|| \cdot ||_{W^{1,p}}$.

58. Theorem (global approximation up to the boundary)
Ω – open, connected, bounded and $\partial\Omega \in C^1$.
Assume $u \in W^{1,p}(\Omega)$, $1 \leqslant p < \infty$. Then there exists a sequence of functions $u_m \in C^\infty(\overline{\Omega})$ $u_m \to u$ in $W^{1,p}(\Omega)$.

59. Extensions of Sobolev functions
$u \in W^{1,p}(\Omega)$

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega \end{cases}$$

60. Theorem
Assume $\Omega \subset \mathbb{R}^n$, open and bounded, $\partial\Omega \in C^1$, $1 \leqslant p < \infty$, if $u \in W^{1,p}(\Omega)$, then there exists a linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that
    (a) $E(u) = u$ on Ω
    (b) $suppEu \subset V$, V – open and bounded, $\Omega \subset\subset V$

(c) $||Eu||_{W^{1,p}(\mathbb{R}^n)} \leqslant C||u||_{W^{1,p}(\Omega)}$
where $C = C(n,p,\Omega,V) > 0$.
Function Eu is called the extension of u on $\mathbb{R}^n$.

61. Theorem
Let $1 \leqslant p < n$, $p^* = \frac{pn}{n-p} > p$, for every $u \in C_0^1(\mathbb{R}^n)$ the inequality holds:
$||u||_{L^{p*}(\mathbb{R}^n)} \leqslant C||Du||_{L^p(\mathbb{R}^n)}$
when $C = C(n,p) > 0$.

62. Theorem (The Sobolev embedding theorem)
$\Omega \subset \mathbb{R}^n$, open and bounded, $\partial\Omega \in C^1$, $1 \leqslant p < n$. Then if $u \in W^{1,p}(\Omega)$ then $u \in L^{p^*}(\Omega)$, where $p^* = \frac{np}{n-p} > p$ and $||u||_{L^{p^*}(\Omega)} \leqslant C||u||_{W^{1,p}(\Omega)}$, $C = c(n,p,\Omega)$.

63. Remarks

- $1 \leqslant p < n$, $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, $p^* = \frac{pn}{n-p}$
- $p > n$, $W^{1,p}(\Omega) \subset C^\alpha(\Omega)$ (Morrey's theorem)
- $u \in W^{1,n}(\Omega) \Rightarrow u \in L^q(\Omega)$, $\forall_{q\geqslant 1}$ ($< \infty$!).

64. Theorem (the Rellich-Kondrachov Compactness Theorem)
$\Omega \subset \mathbb{R}^n$, open and bounded, $\partial\Omega \in C^1$, $1 \leqslant p < n$. Then $W^{1,p} \subset\subset L^q(\Omega)$, $\forall_{q\in[p,p^*)}$, $p^* = \frac{pn}{n-p}$, that is:

(a) $||u||_{L^q(\Omega)} \leqslant C||u||_{W^{1,p}(\Omega)}$
(b) from every sequence bounded in $W^{1,p}(\Omega)$ we can choose a subsequence which is convergent in $L^q(\Omega)$

65. Another version of the R-K theorem
Ω – open and bounded in $\mathbb{R}^n$, $1 \leqslant p < n$, then $W_0^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for $q \in [1, p^*)$.

66. Remark
The condition (b) is equivalent to:

- the unit ball (open) in $W^{1,p}(\Omega)$ is precompact in $L^q(\Omega)$
- the closed unit ball in $W^{1,p}(\Omega)$ is compact in $L^q(\Omega)$
- the open unit ball in $W^{1,p}(\Omega)$ is totally bounded in $L^q(\Omega)$
- B – the unit ball in $W^{1,p}(\Omega)$:
$B = \{u \in W^{1,p}(\Omega) : ||u||_{W^{1,p}(\Omega)} < 1\}$
- $\forall_{\varepsilon>0}\exists_{m(\varepsilon)\in\mathbb{N}}$ and functions $f_1, f_2, \ldots, f_{m(\varepsilon)} \in L^q(\Omega)$ such that $B \subset \bigcup_{j=1}^{m(\varepsilon)} K(f_i,\varepsilon)$
$K(f_i,\varepsilon) = \{f \in L^q(\Omega) : ||f_i - f||_{L^q(\Omega)} < \varepsilon\}$.

67. Arzela-Ascoli Theorem
From every bounded sequence in B_ε, we can extract a subsequence which is uniformly convergent (on every compact set $K \subset \mathbb{R}^n$). Then the subseqence is also convergent in $L^1(V)$:

- B_ε is precompact in $L^1(V)$
- B_ε is totally bounded in $L^1(V)$.

68. Operatory różniczkowe $Lu =$
$= -\sum_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u$
lub
$= -\sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u$
Współczynniki a_{ij}, b_i, $c \in L^\infty(\Omega)$, $u \in C^2(\Omega)$ (na razie).
$\Omega \subset \mathbb{R}^n$ – ograniczony
$\partial\Omega$ – klasy C^1
Zakładamy, że $a_{ij} = a_{ji}$.

69. Jednostajna eliptyczność
L jest jednostajnie eliptyczny $\Leftrightarrow \exists_{\theta>0}$: $\forall_{\xi\in\mathbb{R}^n}$ i dla prawie wszystkich $x \in \Omega$ $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2$.

70. Modelowy problem – zagadnienie Dirichleta

$$(***) = \begin{cases} Lu = f & \text{w } \Omega \subset \mathbb{R}^n \ (f \text{ np.} \in L^2(\Omega)) \\ u_{|\partial\Omega} = 0 \end{cases}$$

Bierzemy $v \in C_0^\infty(\Omega)$ – dowolna, $vLu = vf$ i całkujemy (pierwszy składnik (1) dodatkowo przez części). Dostajemy $B[u,v]$.

71. Forma kwadratowa stowarzyszona z operatorem
$B[u,v] = \sum_{i,j=1}^n \int_\Omega a_{ij}(x)u_{x_i}v_{x_j}dx + \sum_{i=1}^n \int_\Omega b_i(x)u_{x_i}vdx + \int_\Omega c(x)uvdx$
$B : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R}$.

72. Uwaga
Jeśli $u \in C^2(\Omega) \cap C(\bar{\Omega})$ spełnia (***) $\Rightarrow$
$B[u,v] = (f,v)_{L^2(\Omega)}\ \forall_{v\in C_0^\infty(\Omega)} \Rightarrow \forall_{v\in W_0^{1,2}(\Omega)}$.

73. Uwaga
Przypuśćmy, że $u \in C^2(\Omega) \cap C(\bar{\Omega})$ spełnia
$B[u,v] = (f,v)_{L^2}\ \forall_{v\in C_0^\infty(\Omega)}$ (ewentualnie $\forall_{v\in W_0^{1,2}(\Omega)}$).
Wtedy $Lu = f$ p.w. w Ω, o ile a_{ij} powiedzmy $\in C^1$ i $Lu = f$ wszędzie, gdy współczynniki $\int_\Omega v(Lu-f)dx = 0\ \forall_{v\in C_0^\infty}$. L oraz f są ciągłe.

74. Słabe rozwiązanie
Funkcja $u \in W_0^{1,2}(\Omega)$ jest słabym rozwiązaniem zagadnienia (***) $\Leftrightarrow B[u,v] = (f,v)_{L^2}\ \forall_{v\in C_0^\infty}$ (równoważnie $\forall_{v\in W_0^{1,2}(\Omega)}$).

75. Uwaga
Tak samo można definiować słabe rozwiązania równania $Lu = f$, $u_{|\partial\omega} = 0$, gdy $f \in (W_0^{1,2}(\Omega))^k$.

76. Uwagi przed lematem L-M
H – przestrzeń Hilberta nad $\mathbb{R}$
$(,)$ – iloczyn skalarny w H
$||u||^2 = (u,u)$ – norma w H
Tw. Riesza: L $f \in H^* \Rightarrow \exists!_{w: f(v)=(w,v)}$
$B : H \times H \to \mathbb{R}$ – dwuliniowa:

- $|B[u,v]| \leqslant \alpha||u||||v||$ – ograniczoność
- $B[u,u] \geqslant \beta||u||^2$ – dla wszystkich $u \in H$ i pewnej stałej $\beta > 0$ – warunek wymuszania

77. Twierdzenie (lemat Laxa-Milgrama)
$B : H \times H \to \mathbb{R}$ – dwuliniowa, ograniczona, spełnia warunek wymuszania. Wtedy:
$\forall_{f \in H^*} \exists!_{u \in H}$ takie że $B[u, v] = f(v)\ \forall v \in H$.

78. Przykład
$H = W_0^{1,2}(\Omega)$
$(u, v)_1 = \int_\Omega \sum_{i=1}^{\infty} u_{x_i} v_{x_i} dx = \int_\Omega \nabla u \nabla v dx$
$(u, v)_2 = \int_\Omega \nabla u \nabla v dx + \int_\Omega uv dx$
Iloczyny $(,)_1$ i $(,)_2$ generują na $W_0^{1,2}$ tę samą topologię (mają równoważne normy).

79. Wniosek
Jeśli $L = -\sum_{i,j=1}^{n} (a_{ij} - u_{x_i})_{x_j} + cu$ – jednostajnie eliptyczny operator ($b_i = 0$) i $c \geqslant 0$, to zagadnienie (***) $Lu = f$, $u \in W_0^{1,2}(\Omega)$ ma dokładnie jedno słabe rozwiązanie $u \in W_0^{1,2}(\Omega)$.

80. Weakly harmonic/weak solution
Ω – open, bounded $\subset \mathbb{R}^n$.
Laplace's equation: $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ (*).
Weak formulation: $\int_\Omega \nabla u \nabla \phi dx = 0\ \forall_{\phi \in C_0^\infty(\Omega)}$.
If $u \in W^{1,2}(\Omega)$ and it satisfies (*), we say that u is weakly harmonic.
u is a weak solution to the Dirichlet problem for the Laplace equation.

81. Theorem (the Weyl lemma)
If $u \in W^{1,2}(\Omega)$ is a weak solution to the Dirichlet problem for the Laplace equation then $u \in C^\infty(\Omega)$ and $\nabla u = 0$ in the classical sense.

82. Remark
The same is true for the Poisson equation, $\Delta u = f \in C^\infty(\Omega)$. Weak solution is classial solution.

83. Local weak solution
Ω – open in $\mathbb{R}^n$.
If $u \in W^{1,2}(\Omega)_{loc}(\Omega)$ and $\int_\Omega \nabla u \nabla \phi dx = 0$ $\forall_{\phi \in C_0^\infty(\Omega)}$, then we say that u is a local weak solution to the Laplace equation (weak solution on every compact subset).

84. Theorem
If $u \in W_{loc}^{1,2}(\mathbb{R}^n)$ is locally weakly harmonic and $|\nabla u| \in L^2(\mathbb{R}^n)$ then u is constant.

85. Caccioppoli inequality
$B(r) \subset\subset B(R) \subset\subset \Omega$, $B(r)$, $B(R)$ – concentric balls. If $u \in W^{1,2}(\Omega)$ is a weak solution to the Laplace equation in Ω then
$\int_{B(r)} |\nabla u|^2 dx \leqslant \frac{16}{(R-r)^2} \int_{B(R) \setminus B(r)} |u - c|^2 dx\ \forall_{c \in \mathbb{R}}$.

86. Heat equation
$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $u = u(x, t)$
$u_t - \Delta_x u = 0$
u – heat/density of some quantity, t – time

87. The Cauchy problem
(the initial value problem) (*)
$u_t - \Delta_x u = 0$ in $\mathbb{R}^n \times (0, \infty)$
$u(x, 0) = g(x) \in \mathbb{R}^n$

88. The classical solution
A function $u \in C^2(\mathbb{R}^n \times (0, \infty) \cap C(\mathbb{R}^n \times [0, \infty])$.

89. The fundamental solution to the heat eq.
The function:

$$E(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp(\frac{-|x|^2}{4t}) & \text{for } x \in \mathbb{R}^n, t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

90. Properties of fundamental solution
   (i) for each $t > 0$ $\int_{\mathbb{R}^n} E(x, t)dx = 1$ (uniform integrality with respect to t)
   (ii) for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ $\lim_{t \to 0^+} \int_{\mathbb{R}^n} \varphi(x)E(x, t)dx = \phi(0)$
   (iii) $E_t - \Delta E = 0$

91. Theorem
Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Define

$$u(x,t) = \begin{cases} \int_{\mathbb{R}^n} E(x-y,t)g(y)dy & \text{for } x \in \mathbb{R}^n, t > 0 \\ g(x) & \text{for } t = 0 \end{cases}$$

Then u is a solution to the Cauchy problem (*).

92. Nonhomogenous heat eq.
(the Cauchy problem)
$u_t(x, t) - \Delta u(x, t) = h(x, t)$, $x \in \mathbb{R}^n$, $t > 0$
$u(x, 0) = g(x)$, $x \in \mathbb{R}^n$
$u = v + w$, where

$$\begin{cases} v_t - \Delta v = 0 \\ v(x,0) = g(x) \end{cases}$$

$$(**) = \begin{cases} w_t - \Delta w = h \\ w(x,0) = 0 \end{cases}$$

$h \in C(\mathbb{R}^n \times (0, \infty)) \cap L^\infty(\mathbb{R}^n \times (0, \infty))$
$g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

93. Theorem
Define a function
$w(x, t) = \int_0^t \int_{\mathbb{R}^n} E(x - y, t - s)h(y, s)dyds$.
Then w is a solution to (**).

94. Theorem
The Cauchy problem
$u_t - \Delta_x u = h$ in $\mathbb{R}^n \times (0, \infty)$
$u(x, 0) = g \in \mathbb{R}^n \times \{0\}$
for $h \in C \cap L^\infty$, $g \in C \cap L^\infty$ ($h \in C_0^2$ for simplicity) has a solution.

95. (Weak) maximum principle for the heat eq.
Let $\Omega_T = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^n$ – open.
Assume that $u \in C^2(\Omega_T) \cap C(\Omega_T) \cap L^\infty(\Omega_T)$ is a solution to the heat eq. $u_t - \Delta u = 0$ in Ω_T then
$\sup_{\Omega_T} u = \sup_{x \in \Omega} u(x,)$
$\inf_{\Omega_T} u = \inf_{x \in \Omega} u(x,)$

96. Uniqueness
The solution to the Cauchy problem
$u_t - \Delta_x u = 0$ $x \in \mathbb{R}^n$, $t > 0$
$u(x, 0) = g(x)$ $x \in \mathbb{R}^n$, $g \in C \cap L^\infty$
is a unique in the class of functions:
$\{C^2(\mathbb{R}^n \times (0, \infty)) \cap C(\mathbb{R}^n \times [0, \infty))$ and $\forall_{T>0} \sup_{\mathbb{R}^n \times [0,T]} u < \infty\}$.

97. Weak solutions to parabolic eqs.

$$(***) = \begin{cases} u_t + Lu = f & \text{in } \omega_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = g & \text{on } \Omega \times \{t=0\} \end{cases}$$

given: $f : \Omega_T \to \mathbb{R}$, $g : \Omega \to \mathbb{R}$
unknown: $u = u(x,t)$
L – second order partial differential operator

98. The divergence form
$L = -\sum_{i,j=1}^n (a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} + c(x,t)u$

99. Non-divergence form
$L = -\sum_{i,j=1}^n a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} + c(x,t)u$
$\Omega \subset \mathbb{R}^n$ – open and bounded
$\Omega_T = \Omega \times [0,T] \in \mathbb{R}^{n+1}$
$u = u(x,t) : \Omega_T \to \mathbb{R}$

100. (Uniformly) parabolic operator L
If there exists a constant $\theta > 0$ such that $\sum_{i,j}^n a^{ij}(x,t)\xi_i\xi_j \geqslant \theta|\xi|^2$ for every $(x,t) \in \Omega_T$, $\xi \in \mathbb{R}^n$.

101. Motivation
Assume that a^{ij}, b^i, $c \in L^\infty(\Omega_T)$, $f \in L^2(\Omega)$, $g \in L^2(\Omega)$ (till the end).
Assume for while that $u = u(x,t)$ is a smooth solution up to the boundary. We define a bilinear form on $W_0^{1,2}(\Omega)$:
$B(u,w;t) = \int_\Omega \sum_{i,j=1}^n a^{ij}(x,t)u_{x_i}w_{x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} + c(x,t)uw]dx$
for $u, w \in W_0^{1,2}(\Omega)$
$u = u(x)$, $w = w(x)$
$(u(\cdot,t),w)_{L^2(\Omega)} + B(u(\cdot,t),w;t) = (f(\cdot,t),w)_{L^2(\Omega)}$

102. Notation

- $\overline{u} : [0,T] \to W_0^{1,2}(\Omega)$
  $\overline{u}(t) \in W_0^{1,2}(\Omega)$, $(\overline{u}(t))(x) = u(x,t)$
- $\overline{f} : [0,T] \to L^2(\Omega)$
  $\overline{f}(t) \in L^2$, $(\overline{f}(t))(x) = f(x,t)$
- $u_t = f - Lu = G(x,t) + (h^{ij}(x,t))_{x_j}$ where
  $G(x,t) = f - cu - \sum_{i=1}^n b^i u_{x_i}$
  $(h^{ij}(x,t))_{x_j} = \sum_{i,j=1}^n (a^{ij}u_{x_i})_{x_j}$
  $f \in L^2(\Omega_T)$
- $u = u(x,t) \in C^\infty(\overline{\Omega_T}) \Rightarrow u \in W^{1,2}(\Omega_T) \Rightarrow u(\cdot,t) \in W^{1,2}(\Omega)$
- $(W_0^{1,2}(\Omega))^* \ni G(\cdot,t) = f(\cdot,t) - c(\cdot,t)u(\cdot,t) - \sum_i b^i(\cdot,t)u_{x_j}(\cdot,t) \in L^2(\Omega)$
- $(W_0^{1,2})^* \ni h^i(\cdot,t)_{x_j} \in L^2(\Omega)$, $i = 1,\ldots,n$
- $u_t(\cdot,t) \in (W_0^{1,2}(\Omega))^*$
- $u_t((\cdot,t),w)_{L^2(\Omega)} \to < u_t(\cdot,t);w >$
  $w \in W_0^{1,2}(\Omega)$, $u_t(\cdot,t) \in (W_0^{1,2}(\Omega))^*$
- if $\overline{u} : [0,T] \to W_0^{1,2}(\Omega)$, then $\overline{u_t} = \frac{d}{dt}\overline{u}$
  $u = u(x,t) \in L^\infty(\overline{\Omega_T})$
  $\overline{u_t} : [0,T] \to (W_0^{1,2}(\Omega))^*$

103. Weak solution to the problem (***)
A function $u \in L^2(0,T;W_0^{1,2}(\Omega))$, $u_t \in L^2(0,T;(w_0^{1,2}(\Omega))^*)$, if
$< \overline{u_t}(\cdot,t),w > + B(\overline{u}(t),w;t) = (\overline{f}(t),w)_{L^2(\Omega}$
for every function $w \in W_0^{1,2}(\Omega)$ and a.e. $t \in [0,T]$.

104. Theorem
If $u \in L^2(0,T;W_0^{1,2}(\Omega))$, $\frac{d}{dt}u = u_t = u' \in L^2(0,T;(W_0^{1,2}(\Omega))^*)$ then $u \in C([0,T],W_0^{1,2}(\Omega))$.

105. Galerkin's method

- $\{w_k\}_{k=1}^\infty$ – orthonormal basis in $L^2(\Omega)$
  orthogonal basis in $W_0^{1,2}(\Omega)$
- Example: $\{w_k\}$ are eigenfunctios of $T = -\Delta : W_0^{1,2}(\Omega) \to L^2(\Omega)$
- We are looking for $\overline{u_m}$ such that a.e. t $\overline{u_m}(t) = \sum_{k=1}^m d_m^k(t)w_k$ and such that $d_m^k(0) = (g,w_k)_{L^2(\Omega)}$ and
  $< \overline{u_m}'(\cdot,t),w_k > + B(\overline{u_m}(\cdot,t);w_k;t) = (\overline{f}(\cdot,t),w_k)_{L^2(\Omega)}$, $k = 1,2,\ldots,m$.

106. Theorem
$\bar{u_m}$ exists.

107. Theorem
There exists a constant $C > 0$, $C = (\Omega, T, \text{coeff. of L})$ such that
$\max_{0\leqslant t\leqslant T} ||\overline{u_m}(\cdot,t)||_{L^2(\Omega)} + ||\overline{u_m}||_{L^2(0,T;W_0^{1,2}(\Omega))} + ||\overline{u_m}'||^*_{L^2(0,T;W_0^{1,2}(\Omega))} \leqslant c(||\overline{F}||_{L^2(0,T,L^2(\Omega))} + ||g||_{L^2(\Omega)})$

108. Theorem
A weak solution to the problem (***) exists.

109. Fact
Weak solution is unique.