Quantal response equilibrium (Jacob K. Goeree, Charles A. Holt and Thomas R. Palfrey)

1. Introduction

- quantal response specifies choice probabilities that are smooth, increasing functions of expected payoffs
- QRE has the property that the choice distributions match the belief distributions used to calculate expected payoffs; stochastic generalization of NE
- assumption of perfect rationality – strong, sometimes surprising predictions; adding errors
- probabilistic choice models (logit, probit) – used to incorporate stochastic elements in to the analysis of individual decisions; QRE – analogous way to model games with noisy players
- formally – QRF maps the vector of expected payoffs from available choices into a vector of choice probabilities that is monotone in the expected payoffs
- QRE imposes the requirement that the beliefs about other players' actions match the equilibrium choice probabilities; QRE requires solving for a fixed point in the choice probabilities, analogous to the NE
- QRE converges to the NE as the quantal response functions become very steep, and approximate best response functions

2. A motivating example: generalized matching pennies

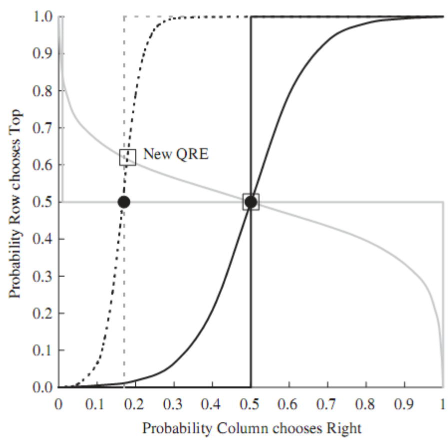

Figure 1 Players' best responses and quantal responses for a generalized matching pennies game

3. Definitions

- $G = (N, S_1, \ldots, S_n, \pi_1 ldots, \pi_n)$ – normal-form game
- $N = \{1, \ldots, n\}$ – the set of players
- $S_i = \{s_{i1}, \ldots, s_{iJ(i)}\}$ – player i's set of strategies
- $S = S_1 \times \ldots \times S_N$ – the set of strategy profiles
- $\pi_i : S_i \to R$ – player i's payoff function
- $\Sigma_i = \Delta^{J(i)}$ – the set of probability distributions over S_i
- $\sigma_i \in \Sigma_i$ – a mixed-strategy, which is a mapping from S_i to Σ_i, where $\sigma(s_i)$ is tthe probability that player i chooses pure strategy s_i
- $\Sigma = \Sigma_1 \times \ldots \times \Sigma_N$ – the set of mixed-strategy profiles
- given a mixed-strategy profile $\sigma \in \Sigma$, player i's expected payoff is $\pi_i(\sigma) = \Sigma_{s \in S} p(s) \pi_i(s)$, where $p(s) = \prod_{i \in N} \sigma_i(s_i)$ is the probability distribution over pure streategy profiles induced by σ.

- P_{ij} – the probability that player i selects strategy j
- Recall that the main idea behind QRE is that strategies with higher expected payoffs are more likely to be chosen, although the best strategy is not necessarily chosen with probability 1. In other words, QRE replaces players' strict rational choice best-responses by smoothed best responses or quantal responses.
- Definition 1
  $P - i : R^{J(i)} \to \Delta^{J(i)}$ is a regular quantal-response function if it satisfies the following four axioms
  - interiority – $P_{ij}(\pi_i) > 0 \ \forall j = 1, \ldots, J(i)$ and $\forall \pi_i \in R^{J(i)}$ (model has full domain)
  - continuity – $P_{ij}(\pi_i)$ is a continuously differentiable function $\forall \pi_i \in R^{J(i)}$ (P_i is non-empty and single-valued; arbitrarily small changes in expected payoffs should not lead to jumps in choice probabilities)
  - responsiveness – $\partial P_{ij}(\pi_i)/\partial \pi_{ij} > 0 \ \forall j = 1 \ldots, J(i)$ and $\forall \pi_i \in R^{J(i)}$ (if the expected payoff of an action increases, the choice probability must also increase)
  - monotonicity – $\pi_{ij} > \pi_{ik}$ implies that $P_{ij}(\pi_i) > P_{ik}(\pi_i) \ \forall j, k = 1, \ldots, J(i)$ (an action with higher expected payoff is chosen more frequently than an action with a lower expected payoff
- Define $P(\pi) = (P_1(\pi_1), \ldots, P_n(\pi_n))$ to be regular if each P_i satisfies the above regularity axioms. Since $P(\pi) \in \Sigma$ and $\pi = \pi(\sigma)$ is defined for any $\sigma \in \Sigma$, $P \circ \sigma$ defines a mapping form Σ into itself.
- Definition 2
  Let P be regular. A regular QRE of the normal-form game G is a mixed-strategy profile σ^* such that $\sigma^* = P(\sigma^*)$.
- Theorem
  There exists a regular QRE of G for any regular P.
  (it follows directly form Brouwer's fixed-point theorem)

4. Empirical implications of regular QRE

- Proposition (Goeree, Holt and Palfrey, 2005)
  In any regular QRE of the asymmetric matching pennies game, Row's probability of choosing Top is strictly increasing in X and Column's probability of choosing Right is strictly decreasing in X.

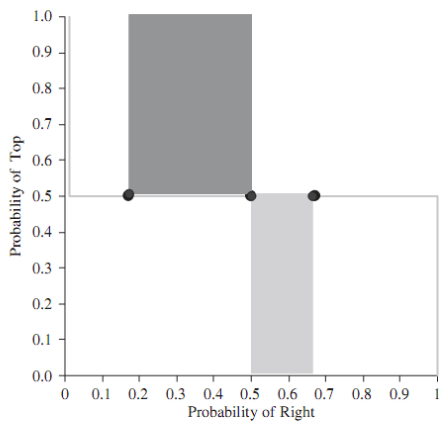

Figure 2 QRE Sets for generalized matching pennies with $X=9$ (dark) and $X=0$ (light)

5. Quantal response equilibrium: a structural definition

- The original definition of QRE (McKelvey and Palfrey, 1995) adopts an approach in the spirit of Harsanyi (1973) and McFadden (1974) whereby the choice probabilities are rationalized by privately observed, mean zero random disturbances to the expected payo?s. These disturbances are assumed to be private information to the players, thereby converting the original game into special kind of game of incomplete information. Any Bayesian equilibrium of this disturbed game is a QRE of the underlying game. The quantal response function generating the QRE is determined by the probability distribution of the random payoff disturbances.
- Thus a smoothed response line can be interpreted to be the (inverse) distribution function of the differences between the disturbances, which has a value of 1/2 when the expected payoffs are equal e.g.
  - if the disturbances are i.i.d. and normally distributed, then the quantal response functions will take the shape of a 'probit' curve,
  - if the i.i.d. disturbances are distributed according to an extreme value distribution, the quantal response functions will have a logistic form

$$p_T = \frac{\exp(\lambda[(X+1)p_R - 1])}{[\exp(\lambda[(X+1)p_R - 1]) + \exp(\lambda[1 - 2p_R])}$$

$$p_R = \frac{\exp(\lambda[1 - 2p_T])}{\exp(\lambda[1 - 2p_T]) + \exp(\lambda[2p_T - 1])}$$

  As the logit precision parameter λ increases, the response functions become more responsive to payoff differences, and the logit response functions converge to the sharp step functions shown in Figure 1.
- interpertation of the disturbance in the structural approach to QRE:
  - one can think of the payoff disturbances as reflecting the effects of unobservable components such as a player's mood or perceptual variations
  - one can think of the players as statisticians, whose objective is to estimate the payoff of each strategy using some unknown set of instruments to perform the estimation
- for general abstract games, a reasonable firrst cut is to suppose that their estimation errors are unbiased
- one can show that the quantal response function generated by i.i.d disturbances will always have the continuity and monotonicity properties of regular quantal response functions, and therefore will lead to regular QRE; if distrurbances are not i.i.d., non-monotonicities are possible

6. Applications: quantal response equilibrium in normal-form games

- In an individual choice problem, the addition of 'noise' spreads out the distribution of decisions around the expected-payo?-maximizing decision. In contrast, expected payo?s in a game depend on other players' choice probabilities, and this interactive element can magnify the effects of noise via feedback effects (examples: coordination game, traveller's dilemma – small amounts of noise can have large effects)
- The QRE has been used to explain 'anomalous' behaviour in a wide variety of games, including signalling games, centipede games, two-stage bargaining, and overbidding in auctions. Moreover it has proven to be quite useful in the analysis of data from political science experiments.

7. Applications: quantal response equilibrium in extensive form games

- In the extensive form QRE, players follow Bayes' rule and calculate expected continuation payoffs based on the QRE strategies of the other players.

- Interiority implies that beliefs are uniquely defined at any information set and for any QRE strategy profile. Therefore issues related to belief-based refinements do not arise, and a quantal response version of sequential rationality follows immediately. When quantal response functions approach best response functions, then the limiting QRE of the extensive form game will select a subset of the sequential equilibria of the underlying game.
- QRE in extensive form games will typically have different implied choice probabilities than would obtain if the same quantal response function were applied to the same game in its reduced normal form.

8. Summary

- The QRE approach to the analysis of games has proven to be a useful generalization of the NE, especially when dealing with 'noisy decisions' made by boundedly rational players and by subjects in experiments.
- It can be extended to allow for learning and cognitive belief formation in one-shot games where learning is not possible.
- This approach provides a coherent framework for analysing an otherwise bewildering array of 'biases' and anomalies in economics.

9. Bibliography

- Anderson, S.K., Goeree, J.K. and Holt, C.A. 2001. Minimum effort coordination games: stochastic potential and the logit equilibrium. Games and Economic Be- havior 34, 177–99.
- Capra, C.M, Goeree, J.K., Gomez, R. and Holt, C.A. 1999. Anomalous behavior in a traveler's dilemma? American Economic Review 89, 678–90.
- Cason, T.N. and Van Lam, M. 2005. Uncertainty and resistance to reform in lab- oratory participation games. European Journal of Political Economy 21, 708–37.
- Goeree, J.K. and Holt, C.A. 2001. Ten little treasures of game theory and ten intuitive contradictions. American Economic Review 91, 1402–22.
- Goeree, J.K. and Holt, C.A. 2005a. An experimental study of costly coordination. Games and Economic Behavior 46, 281–94.
- Goeree, J.K. and Holt, C.A. 2005b. An explanation of anomalous behavior in models of political participation. American Political Science Review 99, 201–13.
- Goeree, J.K., Holt, C.A. and Palfrey, T.R. 2002. Quantal response equilibrium and overbidding in private-value auctions. Journal of Economic Theory 104, 247–72.
- Goeree, J.K., Holt, C.A. and Palfrey, T.R. 2003. Risk averse behavior in asymmetric matching pennies games. Games and Economic Behavior 45, 97–113.
- Goeree, J.K., Holt, C.A. and Palfrey, T.R. 2005. Regular quantal response equilib- rium. Experimental Economics 8, 347–67.
- Guarnaschelli, S., McKelvey, R.D. and Palfrey, T.R. 2000. An experimental study of jury decision rules. American Political Science Review 94, 407–23.
- Haile, P., Hortacsu, A. and Kosenok, G. 2006. On the empirical content of quantal response equilibrium. Working paper, Yale School of Management, Yale University.
- * Harsanyi, J. 1973. Games with randomly disturbed payoffs: a new rationale for mixed strategy equilibrium. International Journal of Game Theory 2, 1–23.
- Levine, D. and Palfrey, T.R. 2007. The paradox of voter participation: an experi- mental study. American Political Science Review 101, 143–58.
- McFadden, D. 1974. Conditional logit analysis of qualitative choice behavior. In Frontiers in Econometrics, ed. P. Zarembka. New York: Academic Press.
- McKelvey, R.D. and Palfrey, T.R. 1995. Quantal response equilibrium for normal form games. Games and Economic Behavior 10, 6–38.
- * McKelvey, R.D. and Palfrey, T.R. 1996. A statistical theory of equilibrium in games. Japanese Economic Review 47, 186–209.
- McKelvey, R.D. and Palfrey, T.R. 1998. Quantal response equilibrium for extensive form games. Experimental Economics 1, 9–41.
- McKelvey, R.D., Palfrey, T.R. and Weber, R. 2000. The effects of payoff magnitude and heterogeneity on behavior in 2x2 games with a unique mixed-strategy equilibrium. Journal of Economic Behavior and Organization 42, 523–48.
- * Ochs, J. 1995. Games with unique, mixed strategy equilibria: an experimental study. Games and Economic Behavior 10, 202–17.