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Population games with attractiveness-driven, instantaneous and delayed strategy choice

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Streszczenie

Population games in systems of agents with attractiveness-driven strategy choice are studied. The attractiveness of the strategy depends on the payoff as well as on the actual popularity in the population. Two-person symmetric games with two strategies as well as games with three strategies are investigated. The attractiveness function is modified by adding additional factors: transcendent factor, rate factor and selection potential. A concept of nonconformist preferences in two-person games with two strategies is introduced. The situation, when agents receive information about the state of the system with delay is also investigated. The dynamical systems describing chosen models, their asymptotic behaviour in time and stability of equilibrium solutions are analyzed. Numerical calculations and plots were performed using Matlab, Octave and Mathematica numerical environments.

Słowa kluczowe

delay differential equations, dynamic systems, population games, social dilemmas, strategic games

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

91 Game theory, economics, social and behavioral sciences
91A Game theory
91A22 Evolutionary games

Tytuł pracy w języku polskim

Gry populacyjne z wyborem strategii określonym przy użyciu funkcji atrakcyjności z uwzględnieniem opóźnienia czasowego

Contents

Introduction

Game theory is a popular and effective tool that enables us to model different types of interactions in diverse scientific branches e.g. biology, social sciences or economics. The evolutionary games formalism is an important mathematical instrument developed by biologists and used to predict population dynamics. Evolutionary games provide a simple framework for describing strategic interactions among large number of players and the evolution of strategies in time. There are many evolutionary game dynamics models used, for instance, replicator dynamics or best response dynamics. Replicator dynamics is one of the most studied evolutionary game dynamics. In replicator dynamics, share of a strategy in a population grows at a rate equal to the difference between the payoff of that strategy and the mean payoff of the population.

Over recent years, a few research activities that use game theory to model various social interactions were conducted. Evolutionary game theory has found several applications to explain the long-term behaviour of systems and the understanding of learning and evolution processes. There are numerous examples where game theory provides deeper understanding of complex social dynamics and still a bigger number of interesting problems to be explored. However, in the most common models too strong assumptions concerning the behaviour of individuals, their motivations and regular mechanisms are usually adopted, like the full rationality of players. It may lead to the results that are contradictory with real and experimental situations.

Nowadays, one of the main aims in the field of modelling interactions using game theory tools is an attempt to find more general models that could better describe the real behaviour of individuals. One of such generalizations of the replicator model was introduced in [1]. The authors proposed replacement of the standard rule of proportional fitness of strategy measured by the payoff from interaction by the more general function called the attractiveness function. The attractiveness of the strategy depends on the payoff as well as on the actual popularity in the population. The parameters of the attractiveness function reflect the different psychological types of agents and refer to the sociological classification introduced by German sociologist Max Weber. Of course, such a simple model still cannot describe social interactions with full precision, however, it can capture more key features of them than the replicator model.

The main objective of this thesis is to study social interactions in large populations using evolutionary game theory tools. I study specific evolution equations that include the attractiveness function mentioned above. I also examine numerically the effect of time delays introduced to the evolution equations on the convergence of solutions to the stationary point. Numerical calculations and plots were performed using Matlab, Octave and Mathematica numerical environments. Below, there is a short summary of the thesis.

Chapter 1 contains a brief introduction and formulation of the model (based on [2]). In the next chapters, I consider two-person symmetric games with two strategies as well as games with three strategies. The relevant evolutionary equations are given, existence and stability of their solutions are studied.

In Chapter 4, I propose modifications of the attractiveness function. It is possible to take into account not only popularity and payoffs from the strategy, but also some other factors. I study the influence on dynamics of transcendent factor, rate of change in popularity of a strategy or so called selection potential that is based on the variance of payoffs. I also consider a concept of nonconformist preferences in two-person games with two strategies. I prove a theorem stating that there exists a unique mixed equilibrium in such case.

It is very likely that decisions may be made by agents who receive a delayed information about some global characteristics of the system. Chapter 5 presents evolutionary games with delays introduced in the attractiveness function. The delays may be added in popularity of the strategy, payoffs or in both of these factors. I check numerically how delay (sometimes called information lag) influences the dynamics. Limit cycles, damped oscillations or a significant decrease in rate of reaching an equilibrium level can be observed for some particular values of the parameters.

Chapter 6 focuses on two-person asymmetric games with two strategies. Criteria excluding existence of a periodic orbit (Bendixson's and Dulac's criteria, recalled in Appendix A.3) are used to show that there are no periodic solutions for such model. A two-person asymmetric game with nonconformist preferences is also considered and the similar result is obtained.

The last chapter includes final conclusions. I sum up the main ideas presented in the thesis and formulate open questions that may become a topic for further work in this field.

My contribution to the thesis are Chapter 4 (presenting modifications of the attractiveness function), Chapter 5 (including numerical investigation of delay evolution equations) and a section concerning nonconformist games in Chapter 6. I have also written the relevant code in Matlab, Octave and Mathematica and performed a lot of varied simulations of two-person symmetric games with two and three strategies. Numerical results presenting trajectories of solutions to non-delay as well as delayed evolutionary equations are included.

I am very grateful to my thesis advisor, prof. Tadeusz Płatkowski, for his support, guidance and helpful suggestions throughout writing the thesis.

Chapter 1

Population games with attractiveness function

This chapter provides an introduction to the evolutionary games played in populations of individuals with complex personality profiles. It is based mostly on [2], [3], [7].

Let us consider an infinite, homogeneous population of individuals (agents). Competition between individuals occurs at each instant of time through pairwise interactions between randomly selected individuals. Each member of the population has the same finite set A of available pure strategies (actions). We use the following notation:

- By $\Delta(A)$ we will denote the $(|A|-1)$-dimensional simplex of $\mathbb{R}^{|A|}$. $K = |A|$ is the number of available strategies.
- $N_i(t)$ is the number of individuals playing strategy i, $i = 1, \ldots, K$. Size N of the population is constant, $N = N_1 + N_2 + \ldots + N_K$.
- By $p(t)$ we will denote the K-dimensional vector whose element $p_i = p_i(t)$ is the frequency of the strategy i (popularity) in the population i.e. $p_i = \frac{N_i}{N}$, $i = 1, 2, \ldots, K$. $p(t)$ is interpreted as a mixed strategy used by all players at time t. Player chooses at time t an action i with probability $p_i(t)$. Vector $p(t)$ is also called the state of the population at time t. $p_i(t)$ is the frequency, so $\sum_{i \in A} p_i(t) = 1$ and $p_i(t) \geq 0$.
- By $\nu_i(t)$ we will denote the expected payoff of an agent playing action i at time t, when the population profile equals $p(t)$, $i = 1, \ldots, K$.

Game theory models of social interactions are often formulated for the general case with n strategies. However, examples usually limit to simple types of games, in which there are two, three or sometimes four actions available.

1.1. Attractiveness function

Many social and biological interactions are based on the process of imitation, where individuals more willingly adopt strategies that are more popular and that probably bring more successes. Therefore, it seems natural to take the popularity of a strategy into account, apart from payoffs.

Let us introduce *the attractiveness function* $u_i(t)$ at time t

$$u_i(t) = p_i^{1-\alpha} \nu_i^{1-\beta} \tag{1.1}$$

where $(\alpha, \beta) \in [0,1] \times [0,1]$, $i = 1, \dots, K$.

The formula (1.1) states that the attractiveness of the strategy i depends on the payoff ν_i (precisely on the mean payoff of the strategy i) as well as on the popularity p_i of this strategy in the population. When $p_i \in (0,1)$, the greater α is, the more influence on the attractiveness of the strategy i the popularity has. The situation is similar for β and ν_i, if all of the payoffs are between 0 and 1. Namely, the greater the value of β is, the more influence on the attractiveness of the strategy i the payoffs have. However, if all payoffs are greater than 1, then with increasing value of β the effect of payoffs on the attractiveness of the strategy i is weaker.

The attractiveness function has been chosen in such a form, because it has numerous important features (that are mentioned in bold) and is well suited to our intuition. The more attractive strategies have an evolutionary advantage in the considered social systems.

The attractiveness function is an **increasing and concave function** of popularity and payoffs of a strategy. Moreover, we get the nonlinear dependence of the attractiveness function on these factors (provided that $\alpha, \beta \neq 1$). The concavity reflects the fact that with increasing attractiveness, the changes are slower. We assume that the attractiveness of a strategy equals zero, if its popularity in the population equals zero.

Moreover, the function (1.1) is the well-known **Cobb-Douglas utility function**. It is widely used in economical problems since it accurately reflects the relation between output and inputs. Parameters α and β describe the reaction of the function u_i on the actual value of popularity and payoffs from the action i. If $\alpha + \beta = 1$, we say that the attractiveness function has constant returns to scale e.g. doubling popularity and payoffs of the strategy will also double its attractiveness. If $\alpha + \beta < 1$, then returns to scale are decreasing, and if $\alpha + \beta > 1$, then returns to scale are increasing.

Parameters α and β are also connected with diverse types of individuals since the attractiveness function can be explained in term of a specific behaviour. Below, we define pure **ideal types of individuals personality profiles** and shortly characterize their behaviour depending on values of α and β. The descriptions are based on the intuitions of German sociologist Max Weber (see [2]).

- *Homo Sociologicus*: $\alpha = 0$, $\beta = 1$
  This individual assesses the attractiveness of the strategy only based on the popularity and he is insensitive to the payoffs of the game ($u_i = p_i$). The information about higher popularity is for him an evidence of success of agents that used this strategy in the past. He imitates the behaviour of the majority.

- *Homo Economicus*: $\alpha = 1$, $\beta = 0$
  This player assesses the attractiveness of the strategy only based on its effectiveness ($u_i = \nu_i$). He does not care about popularity of the strategy.

- *Homo Afectualis*: $\alpha = \beta = 0$
  This individual is maximally sensitive and flexible to changes of factors p_i and ν_i ($u_i = \nu_i p_i$). Dependence of the attractiveness function on popularity and payoffs is linear. The proper evolutionary equation reduces to the standard replicator equation, this is shown in Section 1.2.

- *Homo Transcendentalis*: $\alpha = \beta = 1$
  Each of actions has the same attractiveness for this player ($u_i = 1$). He is not interested in effectiveness or popularity of his behaviour and he is insensitive to theirs changes. He takes into account some other values e.g. aesthetic, ethical issues. Conscience or

other rules are more important for him, he does not reckon with popularity and payoffs from the strategy. He is maximally insensitive and inflexible to the changes of factors p_i and ν_i.

All of the remaining values of parameters α and β reflect the transitive types of personalities. It is very likely that parameters α and β are specific for different types of activity and may change during the life of individual. Their values may depend on social roles or status of the agent as well as on the demographic or cultural characteristics. It may be worth to focus on the "central" model, where $\alpha = \beta = \frac{1}{2}$, from which individuals may deviate to extreme homotypes described above.

1.2. Balance conservation equation

As an alternative to the equilibrium approach, evolutionary game dynamics propose a dynamic updating choice of the strategies. Players myopically update their behaviour in response to the current situation in the population. Strategies that give players higher payoffs and that are more popular increase its share in population through the mechanism of imitation. The rate of change of the actual population state at a given instant of time is assumed to depend on the state of the population at the same time.

Let us consider the following *balance conservation equation* (see [2])

$$\dot{p}_i(t) = \sum_{j \neq i} [p_j r_j p_j^i - p_i r_i p_i^j] \tag{1.2}$$

where p_j^i is the probability that an agent who plays strategy j changes it to strategy i, $i = 1, \ldots, K$.

Assuming that the agent who plays strategy i updates his choice according to the Poisson process with arrival rate r_i, we model the corresponding stochastic process as a deterministic flow. We also assume that p_j^i is proportional to the attractiveness u_i of the strategy i i.e. $p_j^i = cu_i$, $c = const$. It means that the strategies with higher attractiveness are more likely to be chosen. Assuming that the arrival rates r_j are constant (independently of the strategy), after straightforward transformations and rescaling time, we get

$$\begin{aligned} \dot{p}_i(t) &= \sum_{j \neq i} [p_j c u_i - p_i c u_j] = c(\sum_{j \neq i} p_j u_i - p_i u_j) = c[u_i(1 - p_i) - p_i \sum_{j \neq i} u_j + p_i u_i - p_i u_i] = \\ &= c(u_i - u_i p_i - p_i u + p_i u_i) = cu(\frac{u_i}{u} - p_i) \end{aligned}$$

where $u := \sum_{j=1}^{K} u_j$, $i = 1, 2, \ldots, K$.

If we assume that $c = 1$, we obtain the below equation

$$\dot{p}_i = u(\frac{u_i}{u} - p_i) \tag{1.3}$$

where $u := \sum_{j=1}^{K} u_j$, $i = 1, 2, \ldots, K$. Note that for $\alpha = 1$, we assume that $u_i(p_i = 0) = 0$.

The evolutionary equations state that the change of p_i is controlled by its relation with the reference function i.e. the normalized attractiveness function $\tilde{u}_i := u_i/u$. The fraction p_i of the strategy i increases if $\tilde{u}_i$ is greater than the actual fraction of the strategy i in the population, and decreases if it is smaller.

The proper combination of parameters α and β, defined by the *sensitivity parameter*

$$s = \frac{1-\beta}{\alpha}$$

describes the evolution of the considered population in time. Moreover, the sensitivity parameter plays an important role in the matching law in the operant response theory of the mathematical psychology (cf. [2] and the references cited therein).

It is worth observing that for Homo Afectualis i.e. $\alpha = 0$, $\beta = 0$ and $u_i = p_i^{1-\alpha}\nu_i^{1-\beta} = p_i\nu_i$, $i = 1, \ldots K$, we get the standard replicator equation

$$\dot{p}_i = u(\frac{u_i}{u} - p_i) = p_i\nu_i - p_i\sum_{j=1}^{k} p_j\nu_j = p_i(\nu_i - \sum_{j=1,\ldots,K} p_j\nu_j).$$

Thus, our model with attractiveness function can be treated as a **generalization of a standard replicator model**.

Critical points of the dynamics (1.3) can be obtained as solutions to the system of $K-1$ equations

$$\frac{u_1}{p_1} = \frac{u_2}{p_2} = \ldots = \frac{u_K}{p_K}$$

After substituting the attractiveness function to the above equations, for $i, j = 1, \ldots, K$, we get

$$\frac{p_i}{p_j} = (\frac{\nu_i}{\nu_j})^s. \tag{1.4}$$

In particular, it means that the stability properties of solutions to the equations (1.3) depend on the combination s of parameters α and β, not on each of them separately.

Chapter 2

Two-person symmetric games with two strategies

2.1. General model

In this chapter, we assume that $K = 2$. At each instant of time agents play a two-person symmetric game with two strategies C and D and the following payoff matrix

	C	D
C	R	S
D	T	P

where all of the payoffs are nonnegative i.e. $R, S, T, P \geq 0$. In a Prisoner's Dilemma game as well as in the other social dilemma games, C stands for cooperation, D for defection, R for reward, S for sucker, T for temptation and P for punishment. We will denote such a matrix by the vector $[R, S, T, P]$.

The above matrix is a normal-form representation of a game in which players move simultaneously (or do not observe the other player's move before making their own) and receive the relevant payoff. Both players have the same actions available. Considered games are symmetric i.e. payoffs from choosing a particular strategy are the same for both players. Payoffs in the matrix are given for a row player. For example, if row player chooses C and column player chooses D, then row player receives payoff S and column player receives payoff T. In asymmetric games, there are two numbers in each cell of the payoff matrix, the first one represents the payoff of the row player, and the second one represents the payoff of the column player. More information on two-person symmetric games with two strategies can be found in Appendix A.1.

According to (1.1), the attractiveness functions for the first and the second strategy respectively have the form

$$u_1 = p_1^{1-\alpha}\nu_1^{1-\beta}$$
$$u_2 = p_2^{1-\alpha}\nu_2^{1-\beta} = (1-p_1)^{1-\alpha}\nu_2^{1-\beta}$$

For $\alpha = 1$, we assume that $u_i(p_i = 0) = 0$, $i = 1, 2$.

Based on (1.3), we can formulate the relevant evolution equations

$$\begin{cases} \dot{p_1} = u(\frac{u_1}{u} - p_1) = u_1 - up_1 \\ \dot{p_2} = u(\frac{u_2}{u} - p_2) = u_2 - up_2 \end{cases} \tag{2.1}$$

where $u = u_1 + u_2$. From the fact that $p_1 + p_2 = 1$, this system of two equations may be reduced to the following evolution equation

$$\begin{aligned} \dot{p_1} &= p_1^{1-\alpha}\nu_1^{1-\beta} - p_1(p_1^{1-\alpha}\nu_1^{1-\beta} + p_2^{1-\alpha}\nu_2^{1-\beta}) = \\ &= (1-p_1)^{1-\alpha}p_1^{1-\alpha}[(1-p_1)^{\alpha}\nu_1^{1-\beta} - p_1^{\alpha}\nu_2^{1-\beta}]. \end{aligned} \tag{2.2}$$

For the payoff matrix given above, the mean payoffs in the population equal

$$\begin{aligned} \nu_1 &= Rp_1 + Sp_2 = Rp_1 + S(1-p_1) = (R-S)p_1 + S \\ \nu_2 &= Tp_1 + Pp_2 = (T-P)p_1 + P. \end{aligned}$$

After substituting ν_1 and ν_2 to (2.2), we get

$$\begin{aligned} \dot{p_1} =& p_1^{1-\alpha}(Rp_1 + S(1-p_1))^{1-\beta} - \\ &- p_1[(p_1)^{1-\alpha}(Rp_1 + S(1-p_1))^{1-\beta} + (1-p_1)^{1-\alpha}(Tp_1 + P(1-p_1))^{1-\beta}]. \end{aligned} \tag{2.3}$$

Homo Sociologicus $\alpha = 0$, $\beta = 1$	$\dot{p_1} = 0$
Homo Economicus $\alpha = 1$, $\beta = 0$	$\dot{p_1} = (Rp_1 + S(1-p_1)) - p_1[(R+T)p_1 + (S+P)(1-p_1)]$
Homo Afectualis $\alpha = \beta = 0$	$\dot{p_1} = p_1(1-p_1)[p_1(R-T-S+P) + (S-P)]$
Homo Transcendentalis $\alpha = \beta = 1$	$\dot{p_1} = 1 - 2p_1$

Table 2.1: Evolution equations for different individual types.

Now, we can study the dynamics in time and search for the critical points (equilibria) of the ordinary differential equation (2.3). In the case of social dilemmas, where C denotes the first, cooperative strategy, p^* is called (asymptotic) level of cooperation in the population. We can easily distinguish *pure equilibria*:

$$p^* = 0 \qquad \text{or} \qquad p^* = 1.$$

In order to find *mixed equilibria*, we substitute $z = \frac{p_1}{1-p_1}$ to (2.3) and get the following equation

$$\dot{z} = W(z) = [\frac{Rz + S}{Tz + P}]^s - z \tag{2.4}$$

where $z \in (0, \infty)$. The mixed equilibria correspond to the critical points of (2.4) in the interval $(0, 1)$. They have the same stability properties as the critical points of (2.3). Therefore, we can study mixed equilibria and their stability in dynamics (2.3) or equivalently (2.4), see the theorem below (cf. [2]).

Theorem 1. *$p^* \in (0, 1)$ is a stationary solution of equation (2.3) if and only if $z^* = \frac{p_1^*}{1-p_1^*}$ is a stationary solution of (2.4). p^* is stable/unstable if and only if z^* is stable/unstable.*

Existence and asymptotic properties of the solutions to the equation (2.4) depend on the sensitivity parameter $s = \frac{1-\beta}{\alpha}$, $\alpha \neq 0$. As mentioned in Chapter 1, this parameter characterizes the personality profile of players. It is worth noting that mixed equilibria remain the same even if we change parameters α and β, as long as we do not change the sensitivity parameter.

Theorem concerning existence, uniqueness and stability of the mixed equilibria for considered two-person symmetric games is given below. Proofs of Theorem 1 and Theorem 2 were conducted in [2].

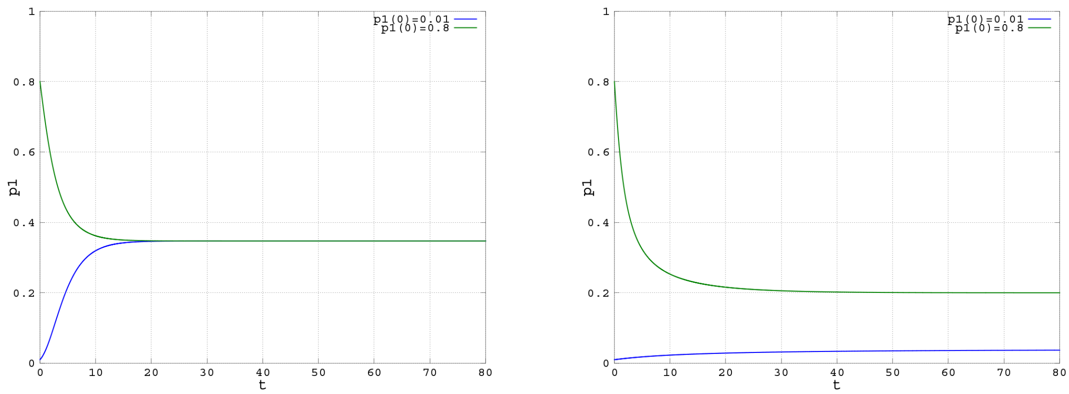
Theorem 2. *For payoff matrix with positive elements $[R, S, T, P]$, we have the following:*

A) For all $0 \leq s < \infty$ there exists at least one mixed equilibrium and at most three mixed equilibria.

B) Let us denote by $B := (1-s)\frac{P}{T} + (1+s)\frac{S}{R}$, $\Delta := B^2 - 4\frac{SP}{RT}$. If $\Delta \leq 0$ or $B \geq 0$ or $ST \geq RP$ then the mixed equilibrium is unique.

C) There exist three mixed equilibria, if and only if $\Delta > 0$, $B < 0$, $U(z_1)U(z_2) < 0$, where $z_{1,2} := \frac{-B \mp \sqrt{\Delta}}{2}$, $U(z) := \ln z + s \ln \frac{Tz+P}{Rz+s}$, $z > 0$.

D) If mixed equilibrium is unique, then it is globally stable in $(0, 1)$. If there exist three mixed equilibria, then the smallest and the greatest are stable, the middle one is unstable.



(a) $[R, S, T, P] = [4, 2, 5, 3]$ (b) $[R, S, T, P] = [38, 1, 40, 11]$

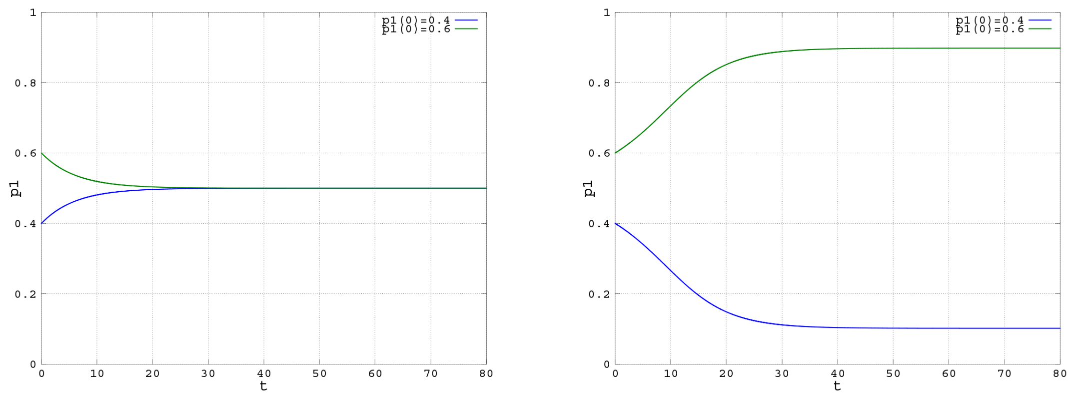
Figure 2.1: The trajectories of solutions to the evolution equation (2.3) for a Prisoner's Dilemma game with parameters $\alpha = 0.2$, $\beta = 0.6$ and different payoff matrices.

2.2. Prisoner's Dilemma game

Prisoner's Dilemma game is one of the most popular and important games in social sciences, studied as the paradigm of evolution of cooperation. The entries in the payoff matrix $[R, S, T, P]$ have to satisfy the condition $T > R > P > S$. If $T > R > P = S$, the game is called *Weak Prisoner's Dilemma*.

In replicator dynamics models, where agents play the one-shot Prisoner's Dilemma game at each instant of time, the only asymptotic state is defection. There are lots of solutions of this dilemma focused on maintaining cooperation in the long term e.g. adding spatial structure to the model, considering iterated interactions, learning by introducing the levels of aspirations etc. Model introduced in this thesis gives the other solution to the dilemma that allows to maintain cooperation.

Theorem 2 (A) and D)) guarantees the existence of a mixed equilibrium for the Prisoner's Dilemma game. This is a very interesting result since under the standard replicator dynamics, there is no mixed equilibrium. In our evolutionary model with attractiveness function, there always exists a stable mixed equilibrium, in which the frequency of cooperators is greater than zero. Under some conditions, there may exist even three mixed equilibria. For instance, for the sensitivity $s = 2$ and entries of the payoff matrix belonging to the set $\{1, \ldots, 40\}$, there are five such matrices $[33, 1, 36, 10]$, $[32, 1, 33, 10]$, $[37, 1, 38, 11]$, $[38, 1, 40, 11]$,$[37, 1, 39, 11]$ (cf. [2]). In Figure 2.1, there are example trajectories for the Prisoner's Dilemma game with parameters $\alpha = 0.2$, $\beta = 0.6$ and different payoff matrices. There is a unique mixed equilibrium for the Prisoner's Dilemma game with payoff matrix $[R, S, T, P] = [4, 2, 5, 3]$ (Fig. 2.1 (a)) and three mixed equilibria for the game with payoff matrix $[R, S, T, P] = [38, 1, 40, 11]$ (Fig. 2.1 (b)).



(a) $\alpha = 0.2$, $\beta = 0.8$, $s = 1 \leq \frac{1+\varepsilon}{1-\varepsilon} = \frac{5}{3}$ (b) $\alpha = 0.2$, $\beta = 0.2$, $s = 4 > \frac{1+\varepsilon}{1-\varepsilon} = \frac{5}{3}$

Figure 2.2: The trajectories of solutions to the evolution equation (2.3) for a coordination game with different sensitivity parameter and payoff matrix $[R, S, T, P] = [2, 1, 1, 2]$.

2.3. Coordination and anti-coordination games

Let us consider a *coordination game* with payoff matrix $[R, S, T, P]$. The payoffs have to satisfy the conditions $R > T$ and $P > S$. We can formulate a stronger result for the uniqueness of the mixed equilibrium than it was stated in Theorem 2 (cf. [2]).

Theorem 3. *For the coordination game i.e.* $R > T$*,* $P > S$ *with* $\varepsilon := \frac{ST}{RP} < 1$*, the mixed equilibrium is unique for* $s \leq \frac{1+\varepsilon}{1-\varepsilon}$ *(see Figure 2.2 (a)).*

For $s > \frac{1+\varepsilon}{1-\varepsilon}$, the previous theorem is not valid. Counterexample with three mixed equilibria is shown in Figure 2.2 (b), two of them are stable.

Let us now define a general *anti-coordination game* with payoff matrix $[R, S, T, P]$. Payoffs satisfy the conditions $T > R > 0$ and $S > P$.

Theorem 4. *For all sensitivities i.e.* $s \in (0, +\infty)$*, there exists a unique mixed equilibrium for the anti-coordination game* $[R, S, T, P]$*, i.e.* $T > R > 0$*,* $S > P \geq 0$*, which is a global attractor of the dynamics (2.3).*

Proofs of Theorem 3 and 4 follow easily from Theorem 2, they can be found in [2].

Chapter 3

Two-person symmetric games with three strategies

We apply earlier approach and introduce the attractiveness function to two-person symmetric games with three strategies i.e. we assume that $K = 3$. We consider complex personality profiles of players and dynamics of imitation, where the player's choice of strategy depends not only on the payoffs, but also on the popularity of the strategy. We focus mainly on the Rock-Paper-Scissors game, but we start from a brief presentation of the general model.

3.1. General model

Let us introduce a general two-person symmetric game with three strategies 1, 2, 3 and the following payoff matrix (cf. [7])

	1	2	3
1	a_{11}	a_{12}	a_{13}
2	a_{21}	a_{22}	a_{23}
3	a_{31}	a_{32}	a_{33}

We assume that $a_{ij} > 0$, $\forall j \in \{1, 2, 3\}$.
Mean payoffs of strategies 1, 2, 3 equal

$$\nu_i(t) = \sum_{j=1}^{3} a_{ij} p_j(t),$$

where $i = 1, 2, 3$. Since $p_1 + p_2 + p_3 = 1$, the dynamical system has the form

$$\begin{cases} \dot{p}_1 = (1-p_1)p_1^{1-\alpha}\nu_1^{1-\beta} - p_1[p_2^{1-\alpha}\nu_2^{1-\beta} + (1-p_1-p_2)^{1-\alpha}\nu_3^{1-\beta}] \\ \dot{p}_2 = (1-p_2)p_2^{1-\alpha}\nu_2^{1-\beta} - p_2[p_1^{1-\alpha}\nu_1^{1-\beta} + (1-p_1-p_2)^{1-\alpha}\nu_3^{1-\beta}] \end{cases} \tag{3.1}$$

After easy transformations and substituting $x := \frac{p_2}{p_1}$ and $y := \frac{p_3}{p_1}$, we get the following mixed equilibria of the system (3.1):

$$\begin{aligned} x &= (\frac{a_{21} + a_{22}x + a_{23}y}{a_{11} + a_{12}x + a_{13}y})^s \\ y &= (\frac{a_{31} + a_{32}x + a_{33}y}{a_{11} + a_{12}x + a_{13}y})^s. \end{aligned} \tag{3.2}$$

Frequencies of the particular strategies are obtained by a straightforward transformation

$$p_1 = \frac{1}{1+x+y}, \qquad p_2 = p_1 x, \qquad p_3 = p_1 y. \tag{3.3}$$

3.2. Rock-Paper-Scissors game

One of the most popular evolutionary games with three strategies is a *Rock-Paper-Scissors game*. This game is a paradigm for a cyclic behaviour in populations (as rock breaks scissors, scissors cut paper, and paper covers rock). We analyze different mathematical properties of the evolutionary dynamics for the Rock-Paper-Scissors game. Most of the results presented below were obtained in [3] and [7].

In our model, asymptotic behaviour of the population change drastically compared to the standard replicator dynamics of the Rock-Paper-Scissors game. The threshold of sensitivity, above which the equilibrium is stable and below which it is unstable, can be found. We also study the relation between the stability of equilibrium and sum of payoffs in the payoff matrix.

3.2.1. Evolutionary scenario

We consider an infinite, homogeneous population of individuals. Players match randomly and interact pairwise. At each instant of time, they play two-person symmetric game with three types of behaviour i.e. R(ock), P(aper), S(cissors), and the following payoff matrix

	R	P	S
R	1	L	V
P	V	1	L
S	L	V	1

where V is the payoff for victory and L is the payoff for loss. Moreover, we assume that $V > 1 > L \geq 0$ and $M := V + 1 + L$. The sum of the entries in rows is constant and equals M.

Let us denote the fraction of players in the population that play strategy R, P and S at time t by $p_1(t)$, $p_2(t)$ and $p_3(t)$ respectively, and the mean payoffs obtained from the relevant strategies by $\nu_1(t)$, $\nu_2(t)$ and $\nu_3(t)$:

$$\begin{aligned}
\nu_1 &= p_1 + Lp_2 + Vp_3 = p_1 + Lp_2 + V(1 - p_1 - p_2)\\
\nu_2 &= Vp_1 + p_2 + Lp_3 = Vp_1 + p_2 + L(1 - p_1 - p_2)\\
\nu_3 &= Lp_1 + Vp_2 + p_3 = Lp_1 + Vp_2 + (1 - p_1 - p_2).
\end{aligned}$$

Note that $\nu_1 + \nu_2 + \nu_3 = M$. Game with such payoffs is called *general Rock-Paper-Scissors game*. In the most popular case, it is assumed that $V = 2$ and $L = 0$. Such a game is called *standard Rock-Paper-Scissors game*.

As before, the attractiveness function for strategies 1, 2 and 3 respectively are the Cobb-Douglas utility functions:

$$\begin{aligned}
u_1 &= p_1^{1-\alpha}\nu_1^{1-\beta}\\
u_2 &= p_2^{1-\alpha}\nu_2^{1-\beta}\\
u_3 &= p_3^{1-\alpha}\nu_3^{1-\beta} = (1 - p_1 - p_2)^{1-\alpha}\nu_3^{1-\beta}.
\end{aligned}$$

For $\alpha = 1$, we assume that $u_i(p_i = 0) = 0$, $i = 1, 2, 3$.

Evolution equations have the form (see (1.2))

$$\begin{cases} \dot{p_1} = u(\frac{u_1}{u} - p_1) = u_1 - up_1 \\ \dot{p_2} = u(\frac{u_2}{u} - p_2) = u_2 - up_2 \\ \dot{p_3} = u(\frac{u_3}{u} - p_3) = u_3 - up_3 \end{cases}$$

where $u = u_1 + u_2 + u_3$. Since $p_1 + p_2 + p_3 = 1$, the system of three equations may be reduced to the below system

$$\begin{cases} \dot{p_1} = (1 - p_1)p_1^{1-\alpha}\nu_1^{1-\beta} - p_1[p_2^{1-\alpha}\nu_2^{1-\beta} + (1 - p_1 - p_2)^{1-\alpha}\nu_3^{1-\beta}] \\ \dot{p_2} = (1 - p_2)p_2^{1-\alpha}\nu_2^{1-\beta} - p_2[p_1^{1-\alpha}\nu_1^{1-\beta} + (1 - p_1 - p_2)^{1-\alpha}\nu_3^{1-\beta}] \end{cases} \tag{3.4}$$

After substituting proper payoff values, we obtain the following evolution equations

$$\begin{cases} \dot{p_1} = p_1^{1-\alpha}(p_1 + Lp_2 + V(1 - p_1 - p_2))^{1-\beta} \\ \qquad - p_1[p_1^{1-\alpha}(p_1 + Lp_2 + V(1 - p_1 - p_2))^{1-\beta} + \\ \qquad + p_2^{1-\alpha}(Vp_1 + p_2 + L(1 - p_1 - p_2))^{1-\beta} \\ \qquad + (1 - p_1 - p_2)^{1-\alpha}(Lp_1 + Vp_2 + (1 - p_1 - p_2))^{1-\beta}] \\ \dot{p_2} = p_2^{1-\alpha}(Vp_1 + p_2 + L(1 - p_1 - p_2))^{1-\beta} \\ \qquad - p_2[p_1^{1-\alpha}(p_1 + Lp_2 + V(1 - p_1 - p_2))^{1-\beta} + \\ \qquad + p_2^{1-\alpha}(Vp_1 + p_2 + L(1 - p_1 - p_2))^{1-\beta} \\ \qquad + (1 - p_1 - p_2)^{1-\alpha}(Lp_1 + Vp_2 + (1 - p_1 - p_2))^{1-\beta}] \end{cases} \tag{3.5}$$

We can study the evolution in time and search for critical points (equilibria) of the above dynamical system. Theorems 5 and 6 recalled below and their proofs can be found in [3].

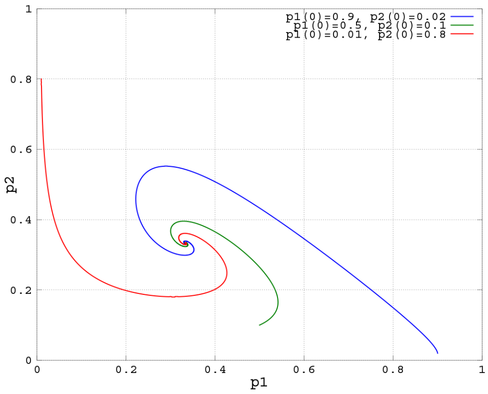
Theorem 5. *For personality types α, $\beta \in (0, 1)$, $s = \frac{1-\beta}{\alpha}$, the critical point $p^* = (\frac{1}{3}, \frac{1}{3})$ is locally asymptotically stable if and only if one of the two following conditions is satisfied:*

- $M \geq 3$
- $2 < M < 3$ *and* $s < \frac{2M}{3-M}$

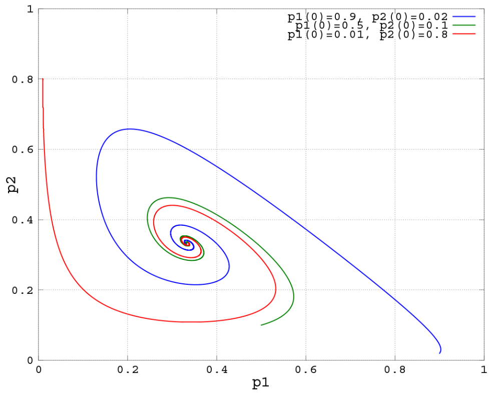
If $2 < M < 3$ and $s > \frac{2M}{3-M}$, then the critical point p^* is unstable (see Figure 3.1)

The first condition states that if the sum of payoffs M is large enough ($M \geq 3$), then the asymptotic stability appears for all sensitivity parameters $s \in (0, \infty)$. The second condition means that for $2 < M < 3$ the stability occurs, when the sensitivity parameter s is small enough i.e. $s < \frac{2M}{3-M}$. Furthermore, if the payoff for victory V reaches the lowest level $V = 1$ (that corresponds to M approaching 2), then the sensitivity cannot be greater than 4 to ensure the stability of the critical point.

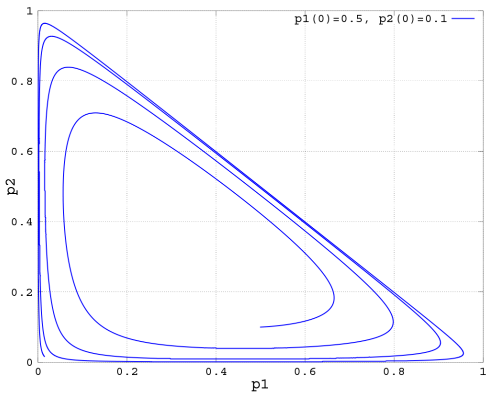
Graphs that illustrates the Theorem 5 are presented in Figure 3.1. For $s = 4\frac{1}{2}$ and $M = 4$, the equilibrium point $p^* = (\frac{1}{3}, \frac{1}{3})$ is stable (Fig. 3.1 (a)). The situation is similar for the smaller value of $M = 2.5$ (Fig. 3.1 (b)). Accordingly to the Theorem 5, for $s = 16$ and $M = 2.5$ the equilibrium point p^* is unstable and the trajectory approaches the boundary of the simplex (Fig. 3.1 (c)).



(a) $\alpha = 0.2$, $\beta = 0.2$, $V = 3$, $L = 0$, $M = 4$
$s = 4 < \frac{2M}{3-M} = 10$



(b) $\alpha = 0.2$, $\beta = 0.2$, $V = 1.5$, $L = 0$, $M = 2.5$
$s = 4 < \frac{2M}{3-M} = 10$



(c) $\alpha = 0.05$, $\beta = 0.2$, $V = 1.5$, $L = 0$, $M = 2.5$,
$s = 16 > \frac{2M}{3-M} = 10$

Figure 3.1: The trajectories of solutions to the evolution equations (3.4) for a Rock-Paper-Scissors game with different parameters and initial conditions.

It is worth noting that for the standard Rock-Paper-Scissors game ($V = 2$ i $L = 0$), the critical point p^* is locally asymptotically stable for all $\alpha, \beta \in (0, 1)$, though, it is only stable in the Lyapunov sense for the classical replicator equations. Introducing the attractiveness function in the evolutionary dynamics has transformed the Lyapunov stability of the critical point into the asymptotic stability.

It is also worthy of observation that for positive sensitivities i.e. for $\beta \neq 1$, the eigenvalues of the Jacobian matrix of the system (3.4) have nonzero imaginary parts (cf. [3]). As expected, the solutions exhibit some kind of cyclic behaviour, which is more visible for smaller values of β. For the critical value $s = \frac{2M}{3-M}$, the eigenvalues are pure imaginary.

In particular, Theorem 5 implies that for $M \in (2, 3)$ and sensitivities large enough the critical point $p^* = (\frac{1}{3}, \frac{1}{3})$ loses its stability. We can study the qualitative behaviour of the critical point using the bifurcation theorem.

Theorem 6. *For $M \in (2, 3)$ and $s = \frac{2M}{3-M}$, the Hopf bifurcation appears. If, additionally,*

$$\beta(V - L)^2 - (V + L)(2 - V - L) < 0,$$

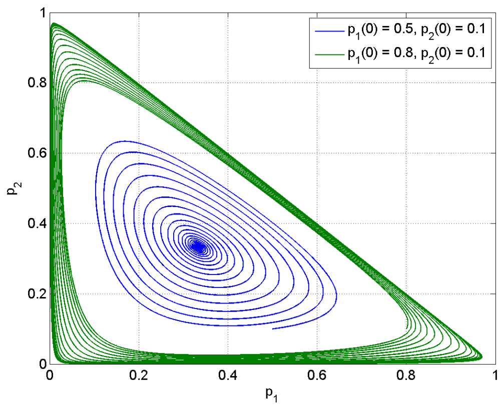
then the bifurcation is supercritical i.e. when point $(\frac{1}{3}, \frac{1}{3})$ *loses its stability, an asymptotically stable limit cycle appears. If*

$$\beta(V-L)^2 - (V+L)(2-V-L) > 0,$$

then the bifurcation is subcritical i.e. when the critical point gains stability, an unstable cycle appears.

3.2.2. Bautin bifurcation in Rock-Paper-Scissors game

As presented in [3], for some values of parameters e.g. $V = 1.4$, $L = 0$, $\alpha = 0.025$ and $\beta = 0.825$, the critical point $p^* = (\frac{1}{3}, \frac{1}{3})$ is stable, but the limit cycle becomes unstable. We observe that a trajectory of solution that starts inside the limit cycle converges to the critical point p^* and the one that starts outside the cycle converges to a larger stable limit cycle, which is situated close to the boundary. It may suggest the occurrence of a secondary fold bifurcation (see Figure 3.2). There is a hypothesis that this phenomenon is related to a Bautin bifurcation. Further investigations in this field may become a topic of the future work.



(a) $\alpha = 0.025$, $\beta = 0.825$

Figure 3.2: The trajectories of solutions to the evolution equations (3.4) for a Rock-Paper-Scissors game with $V = 1.4$ and $L = 0$.

Chapter 4

Modifications of the attractiveness function

4.1. General attractiveness function

The attractiveness function introduced in Chapter 1 (1.1) can be extended and generalized. Apart from popularity and payoffs, we can take other factors into account. General attractiveness function can better reflect different psychological types of individuals since it includes other important factors that may have an impact on agent's choice.

Let us assume that f_i, k_i, l_i, $\gamma \in \mathbb{R}^+$ are constants, $v_i = \frac{D_i^2}{\nu_i^2}$ and D_i^2 is the variance of payoffs, $i = 1, \ldots, K$. We consider the following additional factors that can be added to the attractiveness function:

- *transcendent factor* $f_i^{1-\beta}$
  It reflects an impact of non-material issues that influence the attractiveness of the strategy i. For instance, strategy C may be treated by players as a better strategy than D since cooperation is very often psychologically perceived as more attractive. In general, people usually prefer to cooperate and behave in an unselfish way. Cooperation and supporting other people may bring them non-material positive benefits.

- *rate factor* $(1 + k_i \frac{dp_i}{dt})$
  This factor includes the rate of popularity change of strategy i. It seems reasonable to consider such a factor in the attractiveness function. For instance, it is very common that the attractiveness of some behaviour may be magnified, when this behaviour is getting more and more popular and is spreading quickly among individuals. An analogous phenomenon may occur in the opposite situation, when an action loses its popularity.

- *selection potential* $(1 + l_i v_i)^\gamma$
  Here, we take into account the variance of payoffs. This factor refers to selection and variability. In biological applications, the variance of payoffs is a measure of differentiation and reproduction success among individuals. Species with a great deal of genetic variability is better adapted to the environmental situation and is more likely to survive than a species with limited variability. In our social model, we can also take into account the variance of payoffs. The greater the variance is, the more attractive such strategy may be for the agents. This factor determines in some extent the potential of population to respond to selection (see [6]). Thus, we call this factor selection potential.

The *general attractiveness function* u_i takes the form

$$u_i = p_i^{1-\alpha}\nu_i^{1-\beta}f_i^{1-\beta}(1+k_i\frac{dp_i}{dt})(1+l_iv_i)^{\gamma} \tag{4.1}$$

where $i = 1, \ldots, K$.

Let us now focus on a two-person game with two strategies i.e. we assume that $K = 2$. We will use the following notation

$$\tilde{\nu_i} = f_i\nu_i, \qquad K_i = 1 + k_i\frac{dp_i}{dt} \qquad \text{and} \qquad L_i = (1+l_iv_i)^{\gamma}$$

where $i = 1, 2$. If we insert the attractiveness function (4.1) into the evolution equation (1.3), then we get

$$\begin{aligned}\dot{p_1} =& p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}K_1L_1 - p_1[p_1^{1-\alpha}\tilde{\nu_1}^{1-\beta}K_1L_1 + (1-p_1)^{1-\alpha}\tilde{\nu}_2^{1-\beta}K_2L_2] = \\ =& p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}L_1 + p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}L_1k_1\dot{p}_1 - p_1^{2-\alpha}\tilde{\nu}_1L_1 - \\ & - p_1^{2-\alpha}\tilde{\nu}_1^{1-\beta}L_1k_1\dot{p}_1 - p_1(1-p_1)^{1-\alpha}\tilde{\nu}_2^{1-\beta}L_2 + p_1\dot{p}_1(1-p_1)^{1-\alpha}\tilde{\nu}_2^{1-\beta}L_2k_2.\end{aligned}$$

After transformations, we obtain

$$\begin{aligned}\dot{p}_1(1 - p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}L_1k_1 + p_1^{2-\alpha}\tilde{\nu}_1^{1-\beta}L_1k_1 - p_1(1-p_1)^{1-\alpha}\tilde{\nu}_2^{1-\beta}L_2k_2) = \\ = p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}L_1 - p_1(p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}L_1 + (1-p_1)^{1-\alpha}\tilde{\nu}_2^{1-\beta}L_2).\end{aligned}$$

Finally, the evolution equation for a two-person game with two strategies takes the following form

$$\dot{p_1} = (1 - p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}L_1k_1 + p_1^{2-\alpha}\tilde{\nu}_1^{1-\beta}L_1k_1)^{-1}[(1-p_1)p_1^{1-\alpha}\tilde{\nu}_1^{1-\beta}L_1 + (1-p_1)^{1-\alpha}\tilde{\nu}_2^{1-\beta}L_2] \tag{4.2}$$

Influence on the dynamics of additional factors added to the attractiveness function is studied in sections below.

4.2. Transcendent factor

The attractiveness function (1.1) with the additional transcendent factor $f_i^{1-\beta}$ has the following form

$$u_i = f_i^{1-\beta}p_i^{1-\alpha}\nu_i^{1-\beta} = p_i^{1-\alpha}(f_i\nu_i)^{1-\beta} \tag{4.3}$$

where $i = 1, \ldots, K$. It is clear to see that taking into account the transcendent factor causes a change of payoff matrix and leads to multiplying payoffs in the i-th row by f_i.

In this section, we investigate the influence of the transcendent factor on the dynamics of our model for games with two strategies as well as for a Rock-Paper-Scissors game and a Weak Iterated Prisoner's Dilemma game.

4.2.1. Games with two strategies

Let us consider a two-person symmetric game with two available strategies C and D, and payoff matrix $[R, S, T, P]$, where $R, S, T, P > 0$. It is clear to see that

$$\begin{aligned}f_1\nu_1 &= p_1Rf_1 + (1-p_1)Sf_1 \\ f_2\nu_2 &= p_1Tf_2 + (1-p_1)Pf_2.\end{aligned}$$

Therefore, including the transcendent factor in the attractiveness function leads to the following payoff matrix

	C	D
C	f_1R	f_1S
D	f_2T	f_2P

Obviously, with change of f_1 and f_2 values, the type of game may change (see Figure 4.1). From this point we will assume that $f_1 > 0$ and $f_2 = 1$, so the payoff matrix is as follows

	C	D
C	f_1R	f_1S
D	T	P

(4.4)

Influence of transcendent factor on equilibrium level

The mixed equilibria correspond to the critical points of the evolution equation (2.3) with payoff matrix (4.4). They are equivalent to the critical points of the following equation

$$\dot{z} = (\frac{Rz + S}{Tz + P})^s f_1^s - z \tag{4.5}$$

where $z = \frac{p1}{1-p_1}$ and $s = \frac{1-\beta}{\alpha}$, see (2.4). We can rewrite this equation as follows

$$\dot{z} = (\frac{z + \frac{S}{R}}{z + \frac{P}{T}})^s (\frac{R}{T} f_1)^s - z.$$

The critical points of the above equation satisfy the following condition

$$(1 + \frac{\frac{S}{R} - \frac{P}{T}}{z + \frac{P}{T}})^s = \frac{z}{(\frac{R}{T} f_1)^s}. \tag{4.6}$$

The expression in brackets on the right hand side of the equation (4.6) is a hyperbola and the left hand side of this equation is a straight line with a slope $(\frac{R}{T} f_1)^{-s}$.

Let us assume that initially $f_1 = 1$ and the number of equlibria does not change while changing the f_1 value. We checked numerically that if initially there was a unique mixed equilibrium, then with increasing f_1, the slope of the straight line becomes flatter and the equilibrium point moves right – the level of cooperation increases. Similarly, with decreasing f_1, the level of cooperation decreases. If initially there were three mixed equilibria, then with increasing f_1 the level of cooperation increases for the greatest and the smallest equilibria and decreases for the middle one. For decreasing f_1 the situation is opposite i.e. the level of cooperation decreases for the greatest and the smallest equilibria and increases for the middle one. For instance, for payoff matrix $[R, S, T, P] = [2, 1, 1, 2]$, sensitivity parameter $s = 4$ and transcendent factor $f_1 = 1$, there are three mixed equilibria $p^*_{z_i} = \frac{z_i}{z_i+1}$, $i = 1, 2, 3$, where $z^*_1 \approx 0.11$ $z^*_2 = 1$ and $z^*_3 \approx 8.79$. For $f_1 = 1.05$, z_i values equal respectively 0.20, 0.51, 12.61.

Results for particular values of sensitivity parameter

For some particular values of the sensitivity parameter s, we can give the necessary conditions for existence of a unique root or exactly three roots of the evolution equation (4.5) inside the interval $(0, 1)$.

- $s = 1$
  In this case, the uniqueness is clear – it results from Theorem 2 (moreover, we get the uniqueness for all $s \leq 1$).

  Under our dynamics with attractiveness function, using the equation (4.5) we obtain

  $$f_1(Rz + S) = (Tz + P)z \Leftrightarrow Tz^2 + z(P - f_1 R) - f_1 S = 0.$$

  There is a unique positive root of this equation

  $$z^* = \frac{f_1 R - P + \sqrt{(f_1 R - P)^2 + 4 f_1 T S}}{2T}.$$

  We can easily conclude that for $f_1 > 1$ and $R > P$ the level of cooperation increases compared to the situation when $f_1 = 1$.

- $s = 2$
  Here, we obtain the equation

  $$(Rz + S)^2 f_1^2 = z(Tz + P)^2$$

  which after easy transformations can be rewritten in the form

  $$z^3 T^2 + z^2(PT - R^2 f_1^2) + z(P^2 - 2RS f_1^2) - S^2 f_1^2 = 0.$$

  Let us apply the *Descartes' rule of signs*. The rule states that the number of positive roots of a polynomial with real coefficients ordered by descending variable exponent is equal to the number of sign differences between nonzero coefficients or is less than this number by a multiple of 2.

  Therefore the necessary conditions for existence of three roots are $PT - R^2 f_1^2 < 0$ and $P^2 - 2RS f_1^2 > 0$.

Similar necessary conditions for the existence of three roots (based on Descartes' rule of signs) can be formulated for every natural s.

Influence of transcendent factor on type of game

It is also worth to check how the transcendent factor value influences the type of game. As it was mentioned before, it is possible that the type of game changes while changing the value f_1.

Let us consider games for which the conditions $R > S$ and $T > P$ are satisfied, e.g., Prisoner's Dilemma game (PD), Snow-Drift game (SD), Stag Hunt game (SH, we assume that $T \neq P$) or Harmony game (HG). The relation between the type of game and the f_1 value is depicted in the figure below.

We can also investigate the influence of the f_1 value on a frequency of agents who play strategy C under the standard replicator dynamics. It can be easily calculated that under the assumption $\frac{f_1 R - T}{P - f_1 S} > 0$, this frequency equals

$$\frac{1}{1 + \frac{f_1 R - T}{P - f_1 S}}.$$

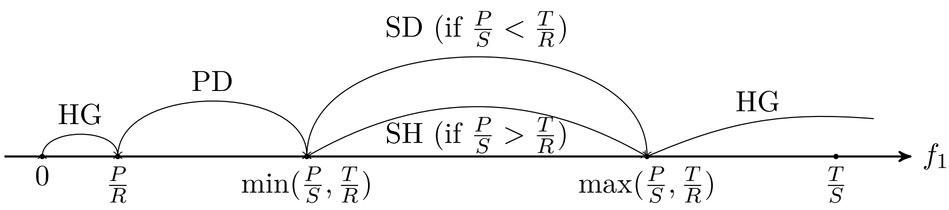


Figure 4.1: Relation between the type of game and the transcendent factor f_1 value.

For instance, if we consider the transcendent factor $f_1 > 1$ for which the type of game does not change, we obtain the following results:

- Snow Drift
  The share of C-players increases in the population unless $T < f_1 R$ (then the type of game changes).

- Anti-coordination game
  There is also an increase in the share of C-players unless $f_1 R > T$.

- Coordination game
  The frequency of C-players decreases unless $f_1 S > P$.

4.2.2. Game with three strategies – Iterated Prisoner's Dilemma game

Let us now focus on the interesting game with three strategies, namely a finitely repeated Prisoner's Dilemma game, which is often called the *Iterated Prisoner's Dilemma game.* Let us denote the number of iterations by m and consider the following strategies (cf. [7]):

- *AllC* – agent plays always strategy C irrespectively from the strategies of the other players

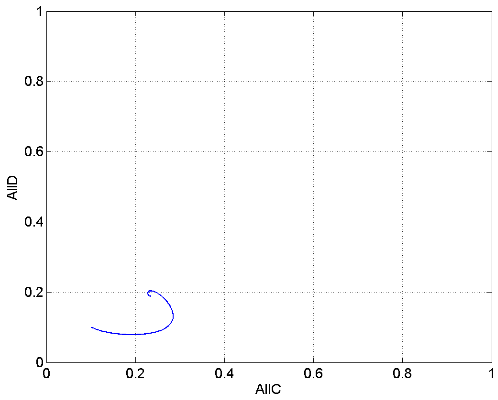
- *AllD* – agent plays always strategy D irrespectively from the strategies of the other players

- *TFT* (Tit For Tat) – agent plays the previous strategy of his opponent, in first round he cooperates.

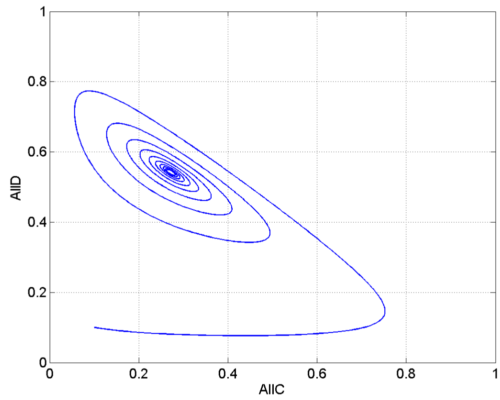
Let us introduce the transcendent factor $f_i^{1-\beta}$ in the attractiveness function, $i = 1, 2, 3$. The payoff matrix for the Iterated Prisoner's Dilemma game with payoffs including transcendent factors has the following form

	AllC	AllD	TFT
AllC	$f_1 Rm$	$f_1 Sm$	$f_1 Rm$
AllD	$f_2 Tm$	$f_2 Pm$	$f_2[T + P(m-1)]$
TFT	$f_3 Rm$	$f_3[S + P(m-1)]$	$f_3 Rm$

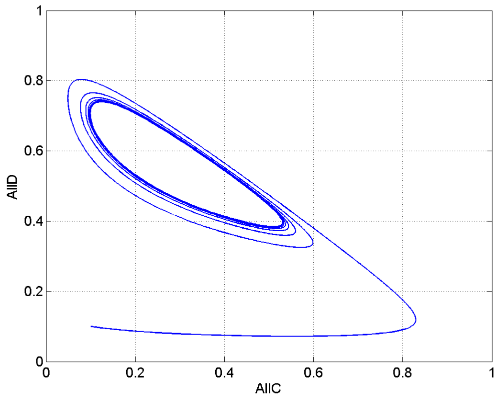
where $T > R > P \geq S$ (if $P = S$ we obtain so called Weak Prisoner's Dilemma game). We also require that $R > (T+S)/2$ to prevent mutual alternate cooperation and defection being more profitable than mutual cooperation.



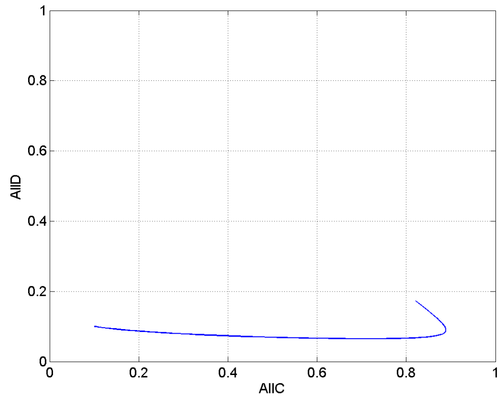
(a) $f_1 = 1$



(b) $f_1 = 1.6$



(c) $f_1 = 1.75$



(d) $f_1 = 1.9$

Figure 4.2: The trajectories of solutions to the evolution equations (3.4) for an Iterated Prisoner's Dilemma with the following parameters: $\alpha = \beta = 0.1$, $[R, S, T, P] = [2, 0, 3, 1]$, $m = 10$, $f_2 = f_3 = 1$, $p_1(0) = p_2(0) = 0.1$ and changing parameter f_1.

Iterated Prisoner's Dilemma game – numerical results

We checked that if we change the parameter f_1 and keep the other parameters f_2 and f_3 equal to 1, then there exists a threshold above which we obtain a limit cycle. Figure 4.2 depicts the trajectories of solutions to the evolution equations (3.4) obtained for the Iterated Prisoner's Dilemma with transcendent preferences for the following values of parameters: $[R, S, T, P] = [2, 0, 3, 1]$, $m = 10$, $\alpha = \beta = 0.1$, $f_2 = f_3 = 1$ and four different f_1. For increasing values of f_1, the oscillations around an equilibrium point occurs. For a narrow interval around 1.75 we observe a limit cycle. For $f_1 = 1.9$ the limit cycle does not exist and the trajectory converges to the stable point, in which the frequency of C-players is higher than 0.8. If we further increase the value of parameter f_1, then the share of $AllC$-players tends to 1.

For comparison and to investigate the speed of approaching equilibrium, we have obtained Figure 4.2 (c) with Dynamo, which is an open-source software that run within Mathematica (see Figure 4.3). The colours in the contour plot represent different speeds of motion under our dynamics with attractiveness function – red means fast and blue means slow.

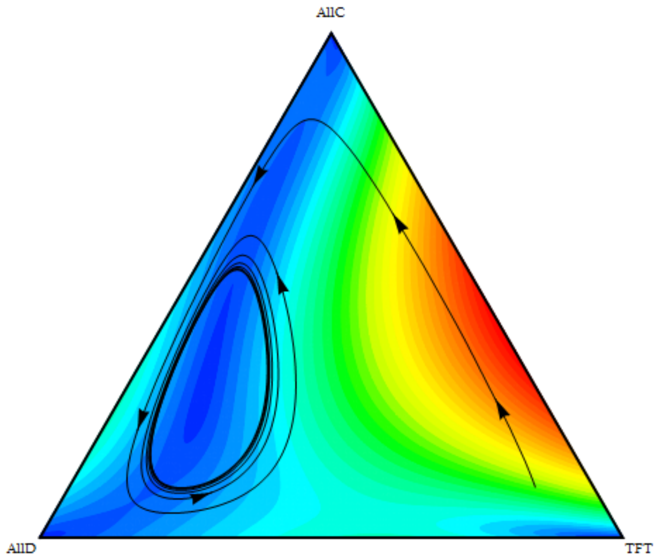


Figure 4.3: The trajectory of solution to the evolution equations (3.4) for an Iterated Prisoner's Dilemma game with the following parameters: $\alpha = \beta = 0.1$, $[R, S, T, P] = [2, 0, 3, 1]$, $m = 10$, $f_1 = 1.75$, $f_2 = f_3 = 1$ and $p_1(0) = p_2(0) = 0.1$.

Weak Iterated Prisoner's Dilemma game – analytical results

Let us return to the general model of two-person symmetric games with three strategies that was introduced in Chapter 3, Section 3.1. We assume that $P = S = 0$ i.e. we consider a Weak Iterated Prisoner's Dilemma game. Based on the results from Chapter 3 (cf. (3.2) and (3.3)), the frequencies of *AllC*, *AllD* and *TFT* players are equal respectively

$$p_1 = \frac{1}{1 + x + y}, \qquad p_2 = p_1 x \qquad \text{and} \qquad p_3 = p_1 y$$

where $x := \frac{p_2}{p_1}$, $y := \frac{p_3}{p_1}$.

Now, we consider the modified payoff matrix including the following transcendent factors:

- $f_1 = f_2 = f_3 = 1$
  According to (3.2), we obtain

$$y = [\frac{Rm + Rmy}{Rm + Rmy}]^s = 1 \qquad \text{and} \qquad x = [\frac{Tm + Ty}{Rm + Rmy}]^s = [\frac{T(m+1)}{2Rm}]^s.$$

  It is worth noting that for increasing ratio $\frac{T}{R}$, the shares of *AllC* and *TFT* players in mixed equilibrium decrease, which is in agreement with intuition.

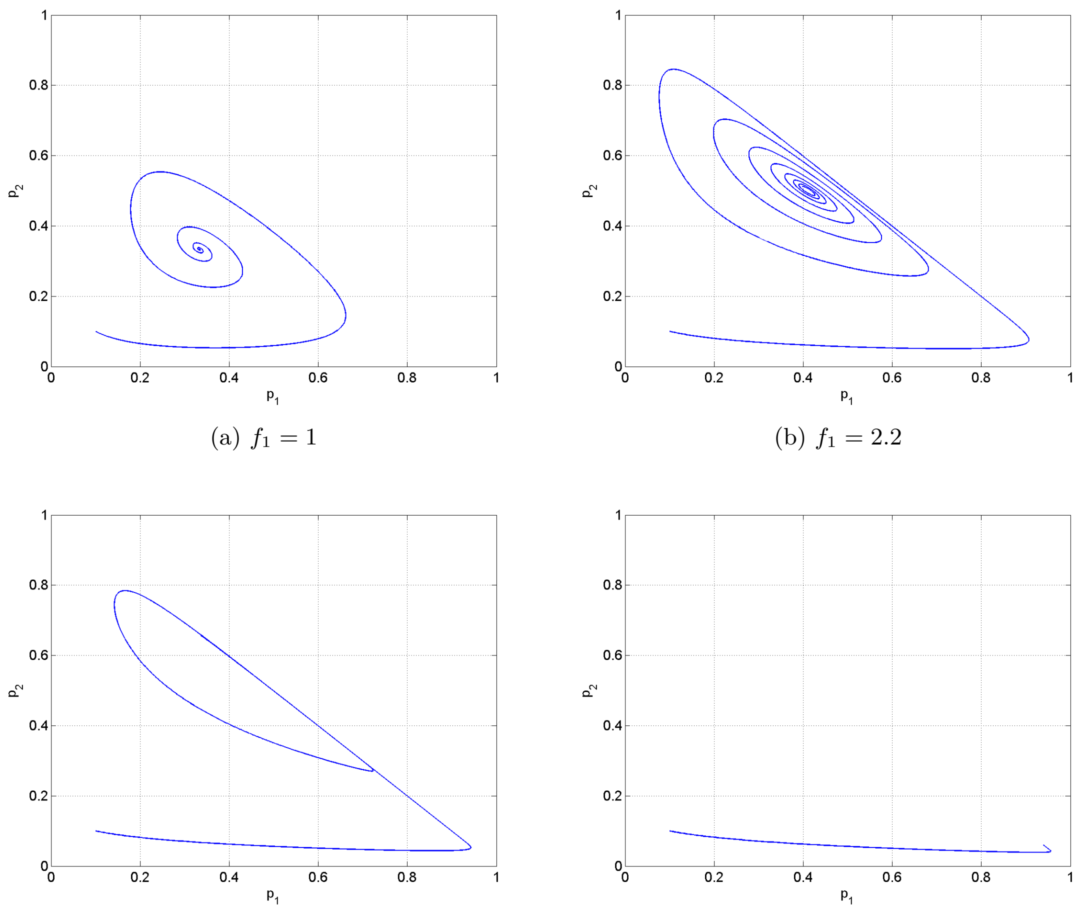
- $f_1 \neq 1$, $f_2 = f_3 = 1$
  In this case

$$y = [\frac{f_1 Rm + f_1 Rmy}{Rm + Rmy}]^s = f_1^s \qquad \text{and} \qquad x = [\frac{T(m+1)}{2f_1 Rm}]^s.$$

As expected, for $f_1 < 1$ the shares of $AllC$ and TFT players decrease in mixed equilibrium, for $f_1 > 1$ they increase.

- $f_2 \neq 1,\ f_1 = f_3 = 1$
  We get
$$y = [\frac{Rm + Rmy}{Rm + Rmy}]^s = 1 \qquad \text{and} \qquad x = [\frac{f_2 T(m+1)}{2Rm}]^s.$$
  Here, the situation is opposite i.e. for $f_2 < 1$ the shares of $AllC$ and TFT players increase in mixed equilibrium, for $f_2 > 1$ they decrease, as it was expected.

- $f_3 \neq 1,\ f_1 = f_2 = 1$
  We obtain
$$y = f_3^s \qquad \text{and} \qquad x = [\frac{T(m + f_3^s)}{Rm(1 + f_3^s)}]^s.$$
  For $f_3 < 1$, the shares of $AllC$ and TFT players decrease in mixed equilibrium, for $f_3 > 1$ they increase. This case is analogous to the second one, when $f_1 \neq 1$, $f_2 = f_3 = 1$.



(a) $f_1 = 1$ (b) $f_1 = 2.2$ (c) $f_1 = 2.6$ (d) $f_1 = 2.8$

Figure 4.4: The trajectories of solutions to the evolution equations (3.4) for a Rock-Paper-Scissors game with the following parameters: $\alpha = \beta = 0.1$, $V = 2$, $L = 0$, $f_2 = f_3 = 1$, $p_1(0) = p_2(0) = 0.1$ and changing parameter f_1.

4.2.3. Game with three strategies – Rock-Paper-Scissors game

Let us consider consider a Rock-Paper-Scissors game with attractiveness function and the following payoff matrix including transcendent factors

	R	P	S
R	f_1	$f_1 L$	$f_1 V$
P	$f_2 V$	f_2	$f_2 L$
S	$f_3 L$	$f_3 V$	f_3

Let us assume that $f_2 = f_3 = 1$, $\alpha = \beta = 0.1$, $V = 2$ and $L = 0$ (standard Rock-Paper-Scissors game). For different values of f_1 we obtain the graphs presented in Figure 4.4. Similarly as it was in the Iterated Prisoner's Dilemma game, if the f_1 value exceeds some threshold, then we obtain a limit cycle. In this case, the limit cycle appears for f_1 in a narrow interval around 2.6 (see also Figure 4.5). However, for $f_1 = 2.8$ the trajectory again quickly approaches a stable equilibrium level. Since the strategies in the Rock-Paper-Scissors game are equivalent, the limit cycle also appears in the analogous intervals for the remaining transcendent factors i.e. when we change f_2 or f_3 value and keep the other parameters equal to 1.

Mathematical explanation of the phenomenon obtained for the Rock-Paper-Scissors game as well as for the Weak Iterated Prisoner's Dilemma game may be a subject of the future research.

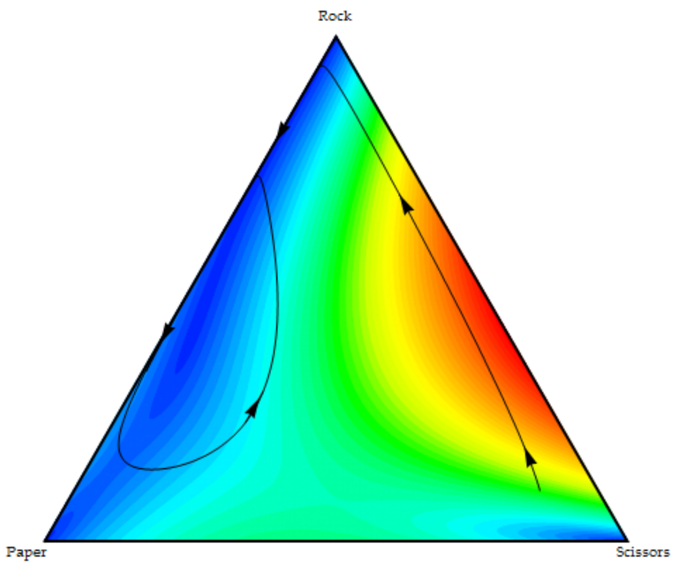


Figure 4.5: The trajectories of solutions to the evolution equations (3.4) for a Rock-Paper-Scissors game with the following parameters: $\alpha = \beta = 0.1$, $V = 2$, $L = 0$ $f_1 = 2.6$, $f_2 = f_3 = 1$ and $p_1(0) = p_2(0) = 0.1$.

4.3. Rate factor

In this section, we focus on the attractiveness function with the rate factor $(1 + k_i \frac{dp_i}{dt})$

$$u_i = (1 + k_i \frac{dp_i}{dt}) p_i^{1-\alpha} \nu_i^{1-\beta} = p_i^{1-\alpha} (f_i \nu_i)^{1-\beta} \tag{4.7}$$

where $i = 1, \ldots, K$. The value of this factor depends on the rate of change in popularity of a strategy i.

Let us consider two-person symmetric games with two strategies. If we include the rate factor in the attractiveness function, then we obtain the following evolution equation

$$\begin{aligned} \dot{p_1} =& p_1^{1-\alpha}\nu_1^{1-\beta}(1 + k_1 \dot{p_1}) - p_1[p_1^{1-\alpha}\nu_1^{1-\beta}(1 + k_1\dot{p_1}) + (1-p_1)^{1-\alpha}\nu_2^{1-\beta}(1 + k_2\dot{p_1})] = \\ =& \dot{p_1}[k_1 p_1^{1-\alpha}\nu_1^{1-\beta} - k_1 p_1(p_1^{1-\alpha}\nu_1^{1-\beta} + k_2 p_1(1-p_1)^{1-\alpha}\nu_2^{1-\beta})] + p_1^{1-\alpha}\nu_1^{1-\beta} - \\ &- p_1(p_1^{1-\alpha}\nu_1^{1-\beta} + (1-p_1)^{1-\alpha}\nu_2^{1-\beta}) \end{aligned}$$

After transformations, we obtain

$$\begin{aligned} \dot{p_1} =& [1 - k_1 p_1^{1-\alpha}\nu_1^{1-\beta} + k_1 p_1(p_1^{1-\alpha}\nu_1^{1-\beta}) - k_2 p_1(1-p_1)^{1-\alpha}\nu_2^{1-\beta}]^{-1} \cdot \\ &\cdot [p_1^{1-\alpha}\nu_1^{1-\beta} - p_1(p_1^{1-\alpha}\nu_1^{1-\beta} + (1-p_1)^{1-\alpha}\nu_2^{1-\beta})] = \\ =& [1 - k_1(1-p_1)p_1^{1-\alpha}\nu_1^{1-\beta} - k_2 p_1(1-p_1)^{1-\alpha}\nu_2^{1-\beta}]^{-1} \cdot \\ &\cdot [(1-p_1)p_1^{1-\alpha}\nu_1^{1-\beta} - p_1(1-p_1)^{1-\alpha}\nu_2^{1-\beta})] \end{aligned}$$

Let us assume that $k_1 = 1$ and $k_2 = 0$ i.e. we include the influence of the rate factor only on the first strategy. The evolution equation takes the form

$$\dot{p_1} = [1 - (1-p_1)p_1^{1-\alpha}\nu_1^{1-\beta}]^{-1} \cdot [(1-p_1)p_1^{1-\alpha}\nu_1^{1-\beta} - p_1(1-p_1)^{1-\alpha}\nu_2^{1-\beta}]. \tag{4.8}$$

If $W(p_1) = (1-p_1)p_1^{1-\alpha}\nu_1^{1-\beta} \in (0,1)$, then the rate factor causes an increase of the rate of convergence to the equilibrium level. It is obvious that $W(p_1)$ is greater than zero, so the only thing we have to check is when

$$(1-p_1)p_1^{1-\alpha}\nu_1^{1-\beta} < 1.$$

This is equivalent to the inequality

$$\nu_1 < (1-p_1)^{\frac{-1}{1-\beta}} p_1^{\frac{\alpha-1}{1-\beta}}.$$

Since $p_1 < 1$, we obtain the following inequalities

$$\nu_1 = (R-S)p_1 + S < R < (1-p_1)^{\frac{-1}{1-\beta}} p_1^{\frac{\alpha-1}{1-\beta}}. \tag{4.9}$$

They are always satisfied when $R \leq 1$.

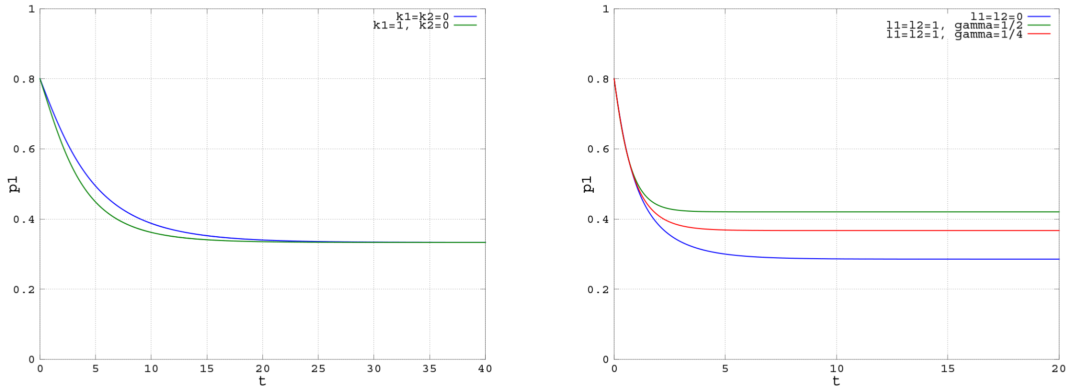
Weak Prisoner's Dilemma game

Let us now consider the Weak Prisoner's Dilemma game with the following payoff matrix

	C	D
C	1	0
D	b	0

(4.10)

where $b > 1$. This payoff matrix was introduced in [5]. The inequalities (4.9) are satisfied for such a payoff matrix since $R = 1$. Hence, we can conclude that for the Weak Prisoner's Dilemma game with payoff matrix (4.10) and the attractiveness function with rate factor $k_1 = 1$ and $k_2 = 0$, we have an increase in the rate of convergence to the equilibrium level (see Figure 4.6 (a)).



(a) Attractiveness function including the rate factor (4.7) with the following parameters: $\alpha = 0.2$, $\beta = 0.8$, $k_2 = 0$, $k_1 = 0$ and $k_1 = 1$. Payoff matrix: $[R, S, T, P] = [1, 0, 2, 0]$.

(b) Attractiveness function including the selection potential (4.11) with the following parameters: $\alpha = 0.5$, $\beta = 0.5$, $l_1 = l_2 = 0$ and $l_1 = l_2 = 1$ (for $\gamma = 0.5$ and $\gamma = 0.25$). Payoff matrix: $[R, S, T, P] = [3, 0, 5, 1]$.

Figure 4.6: The trajectories of solutions to the evolution equations (2.1) with attractiveness function including additional factors for a Weak Prisoner's Dilemma game.

4.4. Selection potential

The attractiveness function including selection potential takes the following form

$$u_i = (1 + l_i v_i)^{\gamma} p_i^{1-\alpha} \nu_i^{1-\beta} f_i^{1-\beta} \tag{4.11}$$

where $i = 1, \dots, K$. We denote by $v_i(x) := \frac{D_i^2}{\nu_i^2}$ the *variability* of strategy i. D_i^2 is the variance of payoffs i.e. $D_i^2 = \sum_j p_j (a_{ij} - \nu_i)^2$, $i = 1, \dots, K$.

We focus on two-person symmetric games with two strategies. The variance of payoffs for strategy C and D respectively equal

$$D_1^2 = p_1(R - (p_1 R + p_2 S))^2 + p_2(S - (p_1 R + p_2 S))^2 = p_1 R^2 + (1 - p_1) S^2 - \nu_1^2$$
$$D_2^2 = p_1(T - (p_1 T + p_2 P))^2 + p_2(P - (p_1 T + p_2 P))^2 = p_1 T^2 + (1 - p_1) P^2 - \nu_2^2.$$

After straightforward transformations, we get

$$v_1(p_1) = \frac{p_1 R^2 + (1 - p_1) S^2}{\nu_1^2} - 1$$
$$v_2(p_1) = \frac{p_1 T^2 + (1 - p_1) P^2}{\nu_2^2} - 1$$

and the evolution equation has the following form

$$\dot{p_1} = (1 - p_1) p_1^{1-\alpha} \nu_1^{1-\beta} (1 + l_1 v_1(p_1))^{\gamma} - p_1 (1 - p_1)^{1-\alpha} \nu_2^{1-\beta} (1 + l_2 v_2(p_1))^{\gamma}. \tag{4.12}$$

Let us assume that $l_1 = l_2 = 1$ and simplify the equation (4.12). We obtain

$$\begin{aligned}\dot{p_1} =& (1 - p_1) p_1^{1-\alpha} \nu_1^{1-\beta} \left(\frac{p_1 R^2 + (1 - p_1) S^2}{(p_1 R + (1 - p_1) S)^2}\right)^{\gamma} - p_1 (1 - p_1)^{1-\alpha} \nu_2^{1-\beta} \left(\frac{p_1 T^2 + (1 - p_1) P^2}{(p_1 T + (1 - p_1) S)^2}\right)^{\gamma} = \\ =& (1 - p_1) p_1^{1-\alpha} \nu_1^{1-\beta-2\gamma} (p_1 R^2 + (1 - p_1) S^2)^{\gamma} - p_1 (1 - p_1)^{1-\alpha} \nu_2^{1-\beta-2\gamma} (p_1 T^2 + (1 - p_1) P^2)^{\gamma}\end{aligned}$$

Let us substitute $z = \frac{p_1}{1-p_1}$. We get

$$\begin{aligned}\dot{z} =& z^{1-\alpha}(z+1)^{\alpha}(Rz+S)^{1-\beta-2\gamma}(z+1)^{\beta+\gamma-1}(Rz^2+S^2)^{\gamma}- \\ &- z(z+1)^{\alpha-1+\beta+\gamma}(zT^2+P^2)^{\gamma}(Tz+P)^{1-\beta-2\gamma} = \\ =&(z+1)^{\alpha-1+\beta+\gamma}z[z^{-\alpha}(Rz+S)^{1-\beta-2\gamma}(Rz^2+S^2)^{\gamma}-(Tz+P)^{1-\beta-2\gamma}(Tz^2+P^2)^{\gamma}].\end{aligned}$$

In order to find the equilibria, we have to solve the following equation

$$z^{-\alpha}(Rz+S)^{1-\beta-2\gamma}(Rz^2+S^2)^{\gamma}-(Tz+P)^{1-\beta-2\gamma}(Tz^2+P^2)^{\gamma}=0.$$

The solutions of the above equation can be expressed in the form

$$z^{-\alpha} = (\frac{Tz+P}{Rz+S})^{1-\beta}[(\frac{Rz+S}{Tz+P})^2(\frac{T^2z+P^2}{R^2z+S^2})]^{\gamma}. \tag{4.13}$$

Let us assume that $S = 0$ and the coefficients α and β satisfy the condition $\alpha + \beta = 1$. Then, the attractiveness function is the classical Cobb-Douglas function with constant returns to scale, multiplied by the selection potential. Under such assumptions, we can rewrite the equation (4.13)

$$1 = (Tz+P)^{\alpha}[z\frac{T^2z+P^2}{(Tz+P)^2}]^{\gamma}. \tag{4.14}$$

For some parameters α and γ we can analytically obtain a precise value of mixed equilibrium using the formula (4.14). Below, we consider the evolution equation for payoff matrix $[3, 0, 5, 1]$ (Prisoner's Dilemma game) and some specific values of parameters α and γ.

- $\alpha = \gamma = \frac{1}{2}$
  After substituting α and γ values and the payoffs to the equation (4.14), we get the quadratic equation
  $$T^2z^2 + z(P^2 - RT) - RP = 0$$
  with $\Delta = P^4 - 2P^2RT + R^2T^2 + 4T^2RP = 496$.
  The unique positive root of this equation is $z^* = \frac{RT-P^2+\sqrt{\Delta}}{2T^2} \approx 0,73$, so $p_1^* \approx \frac{0.73}{1.73} \approx 0.42$.

- $\alpha = \frac{1}{2}$, $\gamma = \frac{1}{4}$
  In this case, we also get the quadratic equation
  $$T^2z^2 + P^2z - R^2 = 0$$
  with $\Delta = P^4 + 4T^2R^2 = 901$.
  The unique positive root of this equation is $z^* = \frac{-P^2+\sqrt{\Delta}}{2T^2} \approx 0.58$, thus, $p_1^* \approx \frac{0.58}{1.58} \approx 0.37$

It is worthy of observation that including the selection potential $(1+l_i\frac{D_i^2}{\nu_i^2})^{\gamma}$ in the attractiveness function may strongly influence the level of cooperation. For the payoff matrix $[3, 0, 5, 1]$, the cooperation level increases significantly compared to the case with "classic" attractiveness function. The equilibrium value for a Prisoner's Dilemma game using the attractiveness function without additional factors (1.1), can be easily calculated and is equal to $p_1^* = \frac{2}{7} \approx 0.29$. This value is significantly smaller than $p_1^* \approx 0.37$ in the first case considered above, when $\alpha = \gamma = \frac{1}{2}$ or $p_1^* \approx 0.42$ in the second case, when $\alpha = \frac{1}{2}$, $\gamma = \frac{1}{4}$. The results are presented in Figure 4.6. Description of the general behaviour of p_1^* in the case of the attractiveness function with selection potential (4.11) needs further numerical computations.

4.5. Nonconformist preferences

So far, in our model, the attractiveness of the strategy has been increasing with increasing popularity. However, some players may have entirely opposite preferences i.e. the strategy may lose its popularity. We will call it the *nonconformist preferences.* In such situations, where people are more likely to make less popular choices to feel elite, our attractiveness function (1.1), introduced in Chapter 1, is inadequate.

Including the concept of nonconformist preferences in the attractiveness function leads to the following modification

$$u_i = (1-p_i)^{1-\alpha}\nu_i^{1-\beta} \tag{4.15}$$

where $i = 1, \ldots, K$.

Below, we prove an interesting theorem for two-person symmetric games with two strategies and nonconformist preferences.

4.5.1. Two-person symmetric games with two strategies

Theorem 7. *There exists a unique mixed equilibrium in two-person symmetric games with nonconformist preferences and positive payoffs.*

Proof. The attractiveness functions take the form

$$u_1 = (1-p_1)^{1-\alpha}\nu_1^{1-\beta} \qquad \text{and} \qquad u_2 = p_1^{1-\alpha}\nu_2^{1-\beta}.$$

We obtain the following evolution equation

$$\begin{aligned}\dot{p_1} &= (1-p_1)^{1-\alpha}\nu_1^{1-\beta} - p_1[(1-p_1)^{1-\alpha}\nu_1^{1-\beta} + p_1^{1-\alpha}\nu_2^{1-\beta}] = \\ &= p_1^{1-\alpha}(1-p_1)^{1-\alpha}[\nu_1^{1-\beta}p_1^{1-\alpha}(1-p_1) - (1-p_1)^{1-\alpha}p_1\nu_2^{1-\beta}]\end{aligned}$$

After substituting $z = \frac{p_1}{1-p_1}$, $z \in (0,1)$, we get

$$\begin{aligned}\frac{\dot{z}}{(z+1)^2} =& z^{1-\alpha}(z+1)^{2(\alpha-1)}[(Rz+S)^{1-\beta}(z+1)^{\beta-\alpha-1}z^{\alpha-1}] - \\ &-(z+1)^{\beta-\alpha-1}z(Tz+P)^{1-\beta}] = P(z)[z^{\alpha-1}(Rz+S)^{1-\beta} - z(Tz+p)^{1-\beta}]\end{aligned}$$

where $P(z) > 0$. Now, we find the critical points of the above equation

$$P(z)[z^{\alpha-1}(Rz+S)^{1-\beta} - z(Tz+p)^{1-\beta}] = 0 \Leftrightarrow (\frac{Rz+S}{Tz+P})^{1-\beta} = z^{2-\alpha}.$$

Let $\tilde{s} = \frac{1-\beta}{2-\alpha}$. We assume that $\alpha \in [0,1]$ and $\beta \in [0,1]$, therefore $\tilde{s} \in [0,1]$. From Theorem 2 B) (Chapter 2) we obtain that $B = (1-s)\frac{P}{T} + (1+s)\frac{S}{R} \geq 0$. Therefore, we can conclude that there exists a unique mixed equilibrium in two-person symmetric game with two strategies. □

Chapter 5

Games with delay

Differential equations with a delayed argument are a very dynamically developing branch of mathematics. Since many problems are still open and for the others there are only partial results, they attract interest of scientists. In natural or social sciences, the equations with delay occur more and more often mainly due to the fact that many processes and phenomena happen with the presence of delay. An individual, an organism or a cell need time to take action after receiving some signal. In biological applications the delay has been introduced many years ago, cf. the Hutchinson model which has been proposed in 1948 as a delayed version of the logistic equation. Differential equations with delay generate infinite dimensional dynamical systems with corresponding phase space that is the functional space (usually the set of continuous functions) and are generally much more difficult to analyze than ordinary differential equations.

Models with delays have been already studied in numerous scientific papers. However, human behavior models with delays have not been thoroughly explored. In our case, model with the delayed attractiveness function better describes the real dynamics of interpersonal relations. People have a natural tendency to analyze their own and/or partner's reactions, which necessarily takes some time.

Basic information about delay differential equations can be found in Appendix A.3

5.1. Attractiveness function with delay

In real situations, it is very common that decisions made by individuals are based not only on present information, but also on a past knowledge. It concerns social as well as economical situations, when it is often difficult to obtain present information. Thus, it seems to be sensible to incorporate delays in our evolutionary model.

In the equations presented in previous chapters, the attractiveness of a strategy i at time t has an instantaneous impact on a rate of change of frequency p_i. An alternative more realistic model would have some delay – the attractiveness acquired at time t will impact the rate of growth τ time later. We introduce delays to the attractiveness function and try to observe the effects on the dynamics of our model. Since the delay evolution equations are very complicated, we focus on numerical investigations. The study presented in this chapter is mostly exploratory and may be a subject of further interest.

Delayed dependence on time in the attractiveness function can be introduced in various ways (cf. 1.1):

- payoffs

$$u_i(t) = (p_i(t))^{1-\alpha}(\nu_i(t-\tau))^{1-\beta}$$

- popularity

$$u_i(t) = (p_i(t-\tau))^{1-\alpha}(\nu_i(t))^{1-\beta}$$

- both popularity and payoffs

$$u_i(t) = (p_i(t-\tau_1))^{1-\alpha}(\nu_i(t-\tau_2))^{1-\beta}$$

where $i = 1, \ldots, K$ and K is the number of available strategies.

In the next sections, results for different types of games are presented. It is worth noting that these results differ significantly depending on the type of game. It is difficult to say something in general about the influence of delays on all games, for instance, in the performed numerical investigations, the oscillations appear only for the delay added to payoffs in the anti-coordination game.

5.2. Two-person symmetric games with two strategies

The evolution equations for a two-person symmetric game with two strategies and attractiveness function with delays in different factors (popularity, payoffs, both popularity and payoffs) are presented below.

A. No delay
Analysis of this equation was presented in Chapter 2 (2.2).

$$\begin{aligned}\dot{p_1} =& (p_1)^{1-\alpha}(Rp_1 + S(1-p_1))^{1-\beta} - p_1[(p_1)^{1-\alpha}(Rp_1 + S(1-p_1))^{1-\beta} + \\ &+ (1-p_1)^{1-\alpha}(Tp_1 + P(1-p_1))^{1-\beta}]\end{aligned}$$

B. Delay in popularity
Popularity of strategy C at time t will impact the rate of growth τ time later.

$$\begin{aligned}\dot{p_1} =& (p_1^\tau)^{1-\alpha}(Rp_1 + S(1-p_1))^{1-\beta} - p_1[(p_1^\tau)^{1-\alpha}(Rp_1 + S(1-p_1))^{1-\beta} + \\ &+ (1-p_1^\tau)^{1-\alpha}(Tp_1 + P(1-p_1))^{1-\beta}]\end{aligned} \tag{5.1}$$

C. Delay in payoffs
Payoff acquired at time t will impact the rate of growth τ time later.

$$\begin{aligned}\dot{p_1} =& (p_1)^{1-\alpha}(Rp_1^\tau + S(1-p_1^\tau))^{1-\beta} - p_1[(p_1)^{1-\alpha}(Rp_1^\tau + S(1-p_1^\tau))^{1-\beta} + \\ &+ (1-p_1)^{1-\alpha}(Tp_1^\tau + P(1-p_1^\tau))^{1-\beta}]\end{aligned} \tag{5.2}$$

D. Delay both in popularity and payoffs
Asymmetric time delays τ_1 and τ_2 are added in both factors (popularity and payoffs).

$$\begin{aligned}\dot{p_1} =& (p_1^{\tau_1})^{1-\alpha}(Rp_1^{\tau_2} + S(1-p_1^{\tau_2}))^{1-\beta} - p_1[(p_1^{\tau_1})^{1-\alpha}(Rp_1^{\tau_2} + S(1-p_1^{\tau_2}))^{1-\beta} + \\ &+ (1-p_1^{\tau_1})^{1-\alpha}(Tp_1^{\tau_2} + P(1-p_1^{\tau_2}))^{1-\beta}]\end{aligned} \tag{5.3}$$

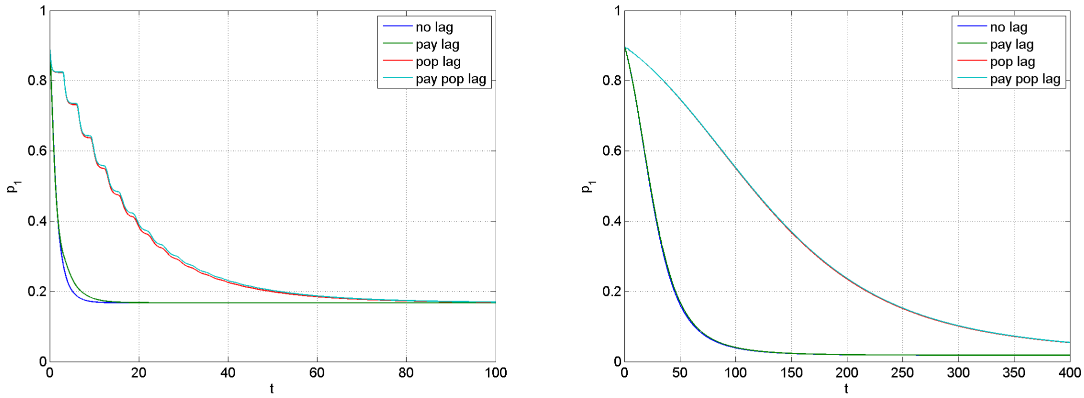
Delay introduced in popularity, payoffs or in both factors causes interesting effects and essential changes of the trajectories of solutions. We focus on the Prisoner's Dilemma game, coordination and anti-coordination games, and we analyze the numerical results obtained for some particular values of parameters.

5.2.1. Prisoner's Dilemma game

In this subsection, our main focus is a Prisoner's Dilemma game with payoff matrix $[R, S, T, P] = [4, 2, 5, 3]$. We consider a case with delay $\tau = 3$. Then the trajectories of solutions to the evolution equation with attractiveness function delayed in payoffs and in both popularity and payoffs are almost the same. Similar behaviour is observed for the attractiveness function delayed in popularity and the non-delayed one.

It is worthy of observation that a significant decrease in the rate of convergence to the equilibrium level for the attractiveness function with delay in popularity and in both factors occurs. In Figure 5.1, the trajectories obtained for $s = 4\frac{1}{2}$ are presented. For greater values of sensitivity the slowdown is more pronounced. The trajectory of solutions to the evolution equation with attractiveness function delayed in popularity or in both factors needs more than 400 units of time to reach the equilibrium level for the sensitivity $s = 10$ (Fig. 5.1 (b)). The delay in payoffs has a minor influence on the rate of convergence.

The notation in figures is as follows: "no lag" – no delays, "pay lag" – delay in popularity, "pop lag" – delay in payoffs, "pay pop lag" – delays of the same value in both factors.



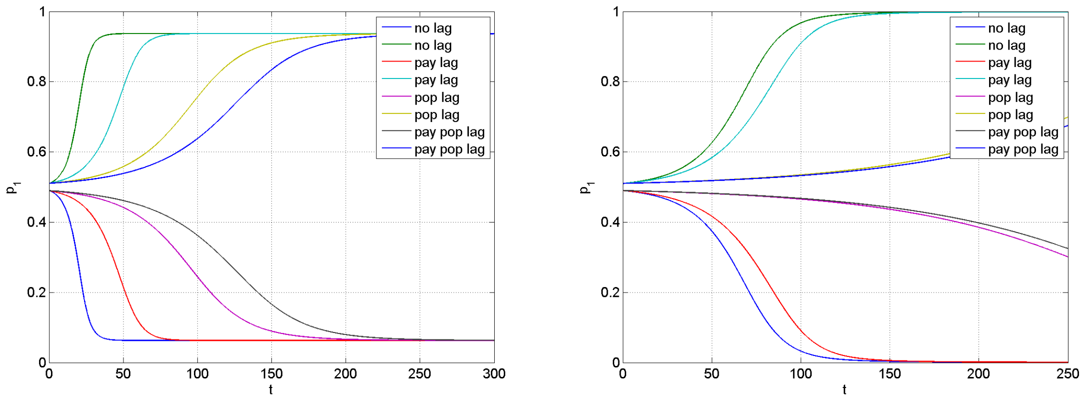
(a) $\alpha = 0.2$, $\beta = 0.1$ (b) $\alpha = 0.02$, $\beta = 0.8$

Figure 5.1: The trajectories of solutions to the evolution equation (2.2) with delays for a Prisoner's Dilemma game with the following parameters: $[R, S, T, P] = [4, 2, 5, 3]$, $\tau = 3$.

5.2.2. Coordination game

Let us consider a coordination game with payoff matrix $[R, S, T, P] = [2, 1, 1, 2]$. Dynamics of the evolution equation without delays for such a game was analyzed in [2]. The author checked that for $s > 3$, there exist two stable mixed equilibria and an unstable mixed equilibrium, which is easy to calculate and equals $p^* = 0.5$.

From numerical investigations, it appears that delays may have an important influence on the dynamics of the evolution equation. Figure 5.2 depicts the relevant trajectories of solutions for the considered coordination game and different sensitivity parameters i.e. $s = 4\frac{1}{2}$ (Fig. 5.2 (a)) and $s = 10$ (Fig. 5.2 (b)). We note that for greater sensitivity parameter, the slowdown of convergence rate for the trajectories of solutions to evolution equations with delay in popularity and in both factors is more significant. Similarly as in the Prisoner's Dilemma game, the delay in payoffs has less influence on the rate of convergence compared to the delay in popularity or delays in both factors.



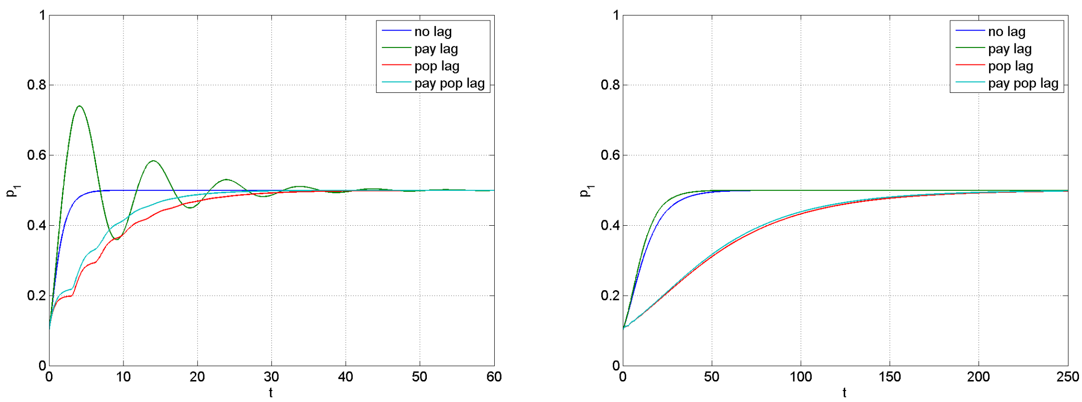
(a) $\alpha = 0.2$, $\beta = 0.1$ (b) $\alpha = 0.02$, $\beta = 0.8$

Figure 5.2: The trajectories of solutions to the evolution equation (2.2) with delays for a coordination game with the following parameters: $[R, S, T, P] = [2, 1, 1, 2]$, $\tau = 3$.

5.2.3. Anti-coordination game

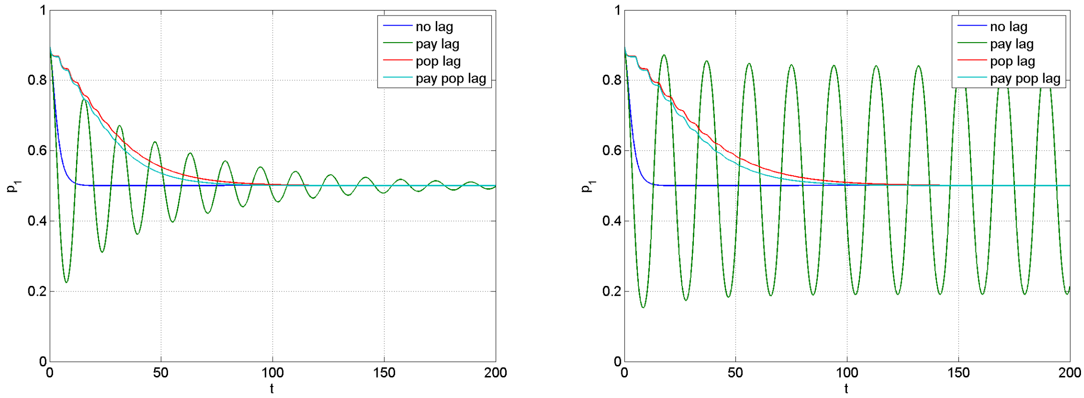
Now, we consider an anti-coordination game with payoff matrix $[R, S, T, P] = [1, 2, 2, 1]$. In the numerical example presented in Figure 5.3, we observe that introducing delays to the attractiveness function causes slower convergence to the equilibrium and may sometimes include the oscillatory behaviour.

Based on Figure 5.3, we note that for the smaller sensitivity parameter $s = 4\frac{1}{2}$ (Fig. 5.3 (a)), the oscillations appear for the trajectory of solution to evolution equation with delay in payoffs. For greater sensitivity $s = 10$ (Fig. 5.3 (b)), the oscillations of this trajectory disappear and the slowdown of convergence is not significant. The behaviour of trajectories for the attractiveness function with delays in popularity or in both popularity and payoffs is similar. The convergence rate is significantly slowed and this is even more pronounced for the greater sensitivity parameter $s = 10$.



(a) $\alpha = 0.2$, $\beta = 0.1$ (b) $\alpha = 0.02$, $\beta = 0.8$

Figure 5.3: The trajectories of solutions to the evolution equation (2.2) with delays for an anti-coordination game with the following parameters: $[R, S, T, P] = [1, 2, 2, 1]$, $\tau = 3$.



(a) $\tau = 4$

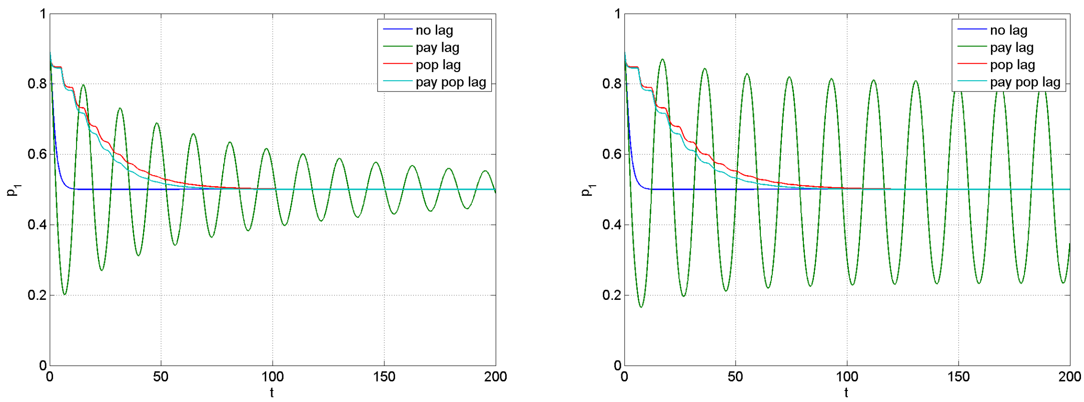
(b) $\tau = 5$

Figure 5.4: The trajectories of solutions to the evolution equation (2.2) for a Snow-Drift game with the following parameters: $[R, S, T, P] = [2, 1, 3, 0]$, $\alpha = 0.02$, $\beta = 0.2$.

Let us now focus on the anti-coordination game called the Snow Drift game. Payoffs of such game satisfy the conditions $T > R > S > P$ (see Appendix A.1).

From numerical investigations for the Snow Drift game with payoff matrix $[R, S, T, P] = [2, 1, 3, 0]$, we can observe that delays added to payoffs in the attractiveness function lead to interesting results. For example, for $\tau = 4$ and sensitivity $s = 40$, damped oscillations appear, see Figure 5.4 (a). However, if the delay is increased to $\tau = 5$, the oscillations do not damp, Figure 5.4 (b) (calculations were performed for $t = 1000$). In Figure 5.5, graphs for the smaller value of sensitivity parameter $s = 8$ are presented. For $\tau = 5$ (Fig. 5.4 (a)), the trajectories still have the form of damping oscillations. However, if we raise the delay to $\tau = 6$ ((Fig. 5.4 (b), the oscillations do not damp.

It is worth noting that for delays in popularity or both in popularity and payoffs, the delays only lead to the slower convergence to the equilibrium level and do not cause the oscillations.



(a) $\tau = 5$

(b) $\tau = 6$

Figure 5.5: The trajectories of solutions to the evolution equation (2.2) for a Snow Drift game with the following parameters: $[R, S, T, P] = [2, 1, 3, 0]$, $\alpha = 0.1$, $\beta = 0.2$.

In Figure 5.6, the approximate relation between values of sensitivity and delays and the oscillatory behaviour of trajectories of solution to the evolution equations with attractiveness function delayed in payoffs (5.2) for the Snow Drift game with payoff matrix $[R, S, T, P] = [2, 1, 3, 0]$ is depicted. Green points denote damping oscillations, while the red ones denote constant oscillations.

Based on Figure 5.6, we can formulate the hypothesis regarding delays added to payoffs in attractiveness function for the considered Snow Drift game.

- For a fixed sensitivity, we can destabilize the equilibrium, if we increase the delay. However, for small values of sensitivity parameter, even large τ does not destabilize the equilibrium.
- For a fixed delay, we can destabilize the equilibrium, if we increase the sensitivity parameter. However, for small values of delay, even large sensitivity does not destabilize the equilibrium.

It should be stressed that the replicator model can be obtained from our model with attractiveness function by assuming the infinite sensitivity i.e. $s \to \infty$. It means that our model may behave in some cases in "less sensitive" way and the equilibrium may be stable, although in the replicator model it is unstable.

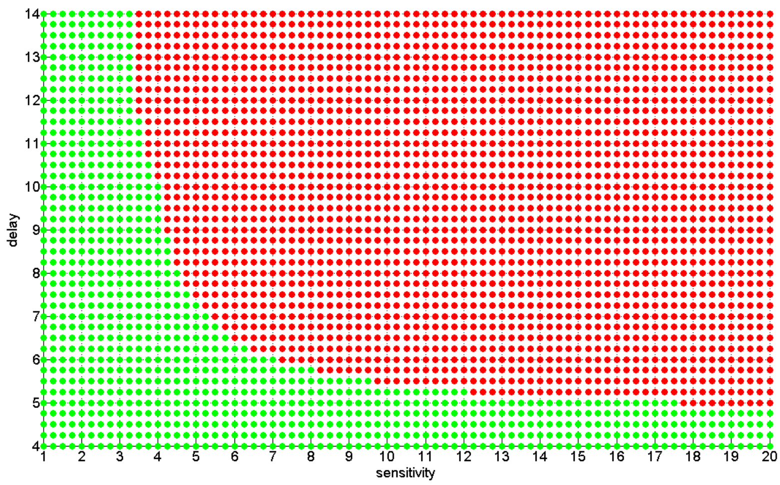


Figure 5.6: Graph presenting the relation between values of sensitivity and delays and the oscillatory behaviour of trajectories of solution to the evolution equations with attractiveness function delayed in payoffs (5.2) for the Snow Drift game with payoff matrix $[R, S, T, P] = [2, 1, 3, 0]$. Green points denote damping oscillations and the red ones denote constant oscillations.

5.3. Rock-Paper-Scissors game

The evolution equations for the Rock-Paper-Scissors game and the attractiveness function with delays in different factors (popularity, payoffs, both popularity and payoffs) are presented below. Let us introduce the following notation: $f^\tau = f(t - \tau)$, $f = f^0 = f(t)$ and $p_3^\tau = 1 - p_1^\tau - p_2^\tau$.

A. No delay
Analysis of this equation was presented in Chapter 3 (see 3.5).

$$\begin{cases} \dot{p_1} =(p_1)^{1-\alpha}(p_1+Lp_2+Vp_3)^{1-\beta} - p_1[(p_1)^{1-\alpha}(p_1+Lp_2+Vp_3)^{1-\beta}+ \\ \qquad + (p_2)^{1-\alpha}(Vp_1+p_2+Lp_3)^{1-\beta} + p_3^{1-\alpha}(Lp_1+Vp_2+p_3)^{1-\beta}] \\ \dot{p_2} =(p_2)^{1-\alpha}(Vp_1+p_2+Lp_3)^{1-\beta} - p_2[(p_1)^{1-\alpha}(p_1+Lp_2+Vp_3)^{1-\beta}+ \\ \qquad + (p_2)^{1-\alpha}(Vp_1+p_2+Lp_3)^{1-\beta} + (p_3)^{1-\alpha}(Lp_1+Vp_2+p_3)^{1-\beta}] \end{cases}$$

B. Delay in popularity
Popularity of strategies at time t will impact the rate of growth τ time later.

$$\begin{cases} \dot{p_1} =(p_1^\tau)^{1-\alpha}(p_1+Lp_2+Vp_3)^{1-\beta} - p_1[(p_1^\tau)^{1-\alpha}(p_1+Lp_2+Vp_3)^{1-\beta}+ \\ \qquad + (p_2^\tau)^{1-\alpha}(Vp_1+p_2+Lp_3)^{1-\beta} + (1-p_1^\tau-p_2^\tau)^{1-\alpha}(Lp_1+Vp_2+p_3)^{1-\beta}] \\ \dot{p_2} =(p_2^\tau)^{1-\alpha}(Vp_1+p_2+Lp_3)^{1-\beta} - p_2[(p_1^\tau)^{1-\alpha}(p_1+Lp_2+Vp_3)^{1-\beta}+ \\ \qquad + (p_2^\tau)^{1-\alpha}(Vp_1+p_2+Lp_3)^{1-\beta} + (1-p_1^\tau-p_2^\tau)^{1-\alpha}(Lp_1+Vp_2+p_3)^{1-\beta}] \end{cases} \tag{5.4}$$

C. Delay in payoffs
Payoff acquired at time t will impact the rate of growth τ time later.

$$\begin{cases} \dot{p_1} =(p_1)^{1-\alpha}(p_1^\tau+Lp_2^\tau+Vp_3^\tau)^{1-\beta} - p_1[(p_1)^{1-\alpha}(p_1^\tau+Lp_2^\tau+Vp_3^\tau)^{1-\beta}+ \\ \qquad + (p_2)^{1-\alpha}(Vp_1^\tau+p_2^\tau+L(1-p_1^\tau-p_2^\tau))^{1-\beta} + (p_3)^{1-\alpha}(Lp_1^\tau+Vp_2^\tau+p_3^\tau)^{1-\beta}] \\ \dot{p_2} =(p_2)^{1-\alpha}(Vp_1^\tau+p_2^\tau+Lp_3^\tau)^{1-\beta} - p_2[(p_1)^{1-\alpha}(p_1^\tau+Lp_2\tau+Vp_3^\tau)^{1-\beta}+ \\ \qquad + (p_2)^{1-\alpha}(Vp_1^\tau+p_2^\tau+Lp_3^\tau)^{1-\beta} + (p_3)^{1-\alpha}(Lp_1^\tau+Vp_2^\tau+p_3^\tau)^{1-\beta}] \end{cases} \tag{5.5}$$

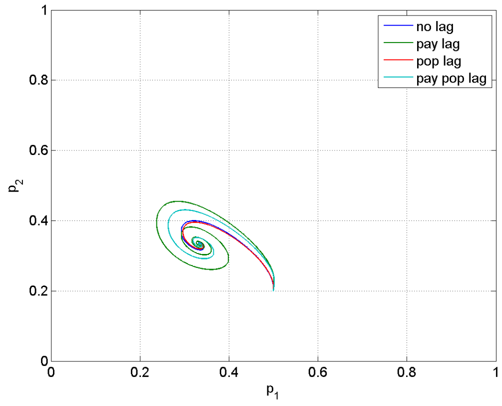
D. Delay both in popularity and payoffs
Asymmetric time delays τ_1 and τ_2 are added in both factors (popularity and payoffs).

$$\begin{cases} \dot{p_1} =(p_1^{\tau_1})^{1-\alpha}(p_1^{\tau_2}+Lp_2^{\tau_2}+Vp_3^{\tau_2})^{1-\beta} - p_1[(p_1^{\tau_1})^{1-\alpha}(p_1^{\tau_2}+Lp_2^{\tau_2}+Vp_3^{\tau_2})^{1-\beta}+ \\ \qquad + (p_2^{\tau_1})^{1-\alpha}(Vp_1^{\tau_2}+p_2^{\tau_2}+Lp_3^{\tau_2})^{1-\beta} + (p_3^{\tau_1})^{1-\alpha}(Lp_1^{\tau_2}+Vp_2^{\tau_2}+p_3^{\tau_2})^{1-\beta}] \\ \dot{p_2} =(p_2^{\tau_1})^{1-\alpha}(Vp_1^{\tau_2}+p_2^{\tau_2}+Lp_3^{\tau_2})^{1-\beta} - p_2[(p_1^{\tau_1})^{1-\alpha}(p_1^{\tau_2}+Lp_2^{\tau_2}+Vp_3^{\tau_2})^{1-\beta}+ \\ \qquad + (p_2^{\tau_1})^{1-\alpha}(Vp_1^{\tau_2}+p_2^{\tau_2}+Lp_3^{\tau_2})^{1-\beta} + (p_3^{\tau_1})^{1-\alpha}(Lp_1^{\tau_2}+Vp_2^{\tau_2}+p_3^{\tau_2})^{1-\beta}] \end{cases} \tag{5.6}$$

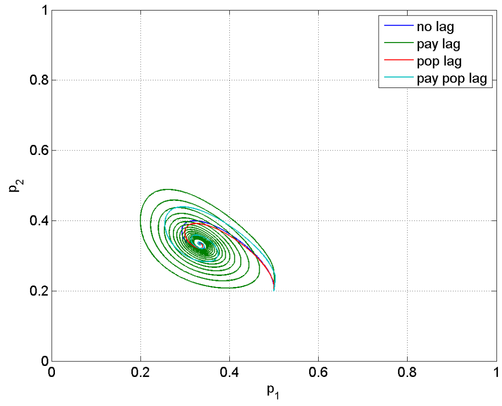
In Figure 5.7, the trajectories of solutions to the evolution equations (3.5) with delays for some particular values of parameters are presented. We assume that $\alpha = \beta = 0.2$ and $V = 2$ and $L = 0$ (standard Rock-Paper-Scissors game).

We start the investigations from $\tau = 0.5$. For this value of delay, all delayed trajectories approaches quickly the stable equilibrium $p^* = (\frac{1}{3}, \frac{1}{3})$. With increasing value of delay, the trajectories for attractiveness function with delays added in popularity or in both factors remain almost unchanged compared to Figure 5.7 (a). For investigated values of delays, these trajectories always approaches the stable point $p^* = (\frac{1}{3}, \frac{1}{3})$. For delay introduced in payoffs, the trajectories of solutions exhibit a very interesting behaviour. For smaller values of delay, we can observe oscillations (Fig. 5.7 (b)) and finally, for $\tau = 0.9$ (Fig. 5.7 (c)), the trajectory approaches a limit cycle. For the greater values of delay, the limit cycle enlarges and approaches the boundary of the simplex.

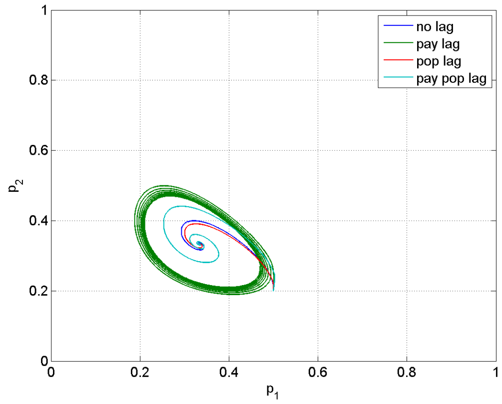
It is worth noting that for general Rock-Paper-Scissors game with different parameters V, L, α and β, the behaviour of trajectories may be different and further research should be conducted. Example graphs are presented in Figure 5.7.



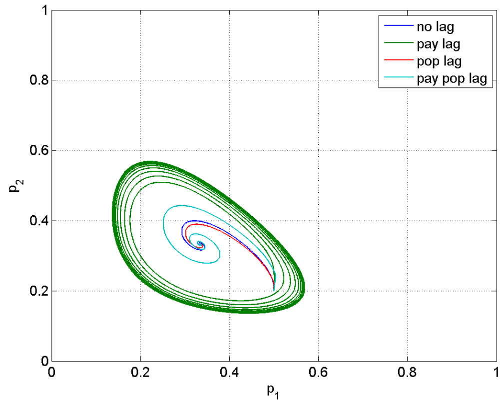
(a) $\tau = 0.5$



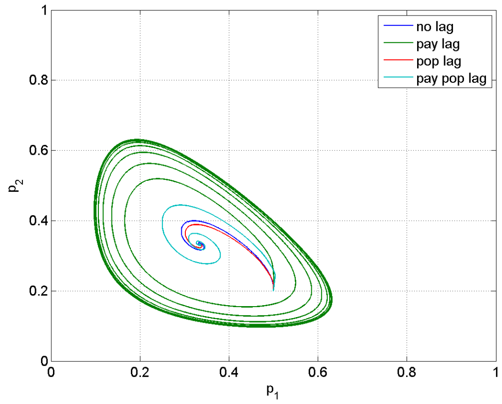
(b) $\tau = 0.8$



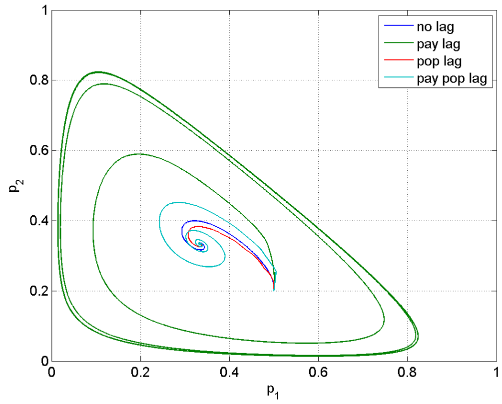
(c) $\tau = 0.9$



(d) $\tau = 1$



(e) $\tau = 1.1$



(f) $\tau = 2$

Figure 5.7: The trajectories of solutions to the evolution equations (3.4) for a Rock-Paper-Scissors game with the following parameters: $\alpha = 0.2$, $\beta = 0.2$, $V = 2$, $L = 0$, $p_1(0) = 0.5$, $p_2(0) = 0.2$.

Chapter 6

Two-person asymmetric games with two strategies

In this chapter, we consider two-person asymmetric games with two strategies. Games are played between members of two populations and players from each population have different personality profiles.

6.1. General model

At first, let us introduce the relevant payoff matrix. We assume that the row player belongs to the population 1 and the column player belongs to the population 2. Members of each of two populations choose between two strategies A and B. The payoff matrix has the following form

$$\begin{array}{c|cc} & \text{A} & \text{B} \\ \hline \text{A} & (a_1, a_2) & (b_1, b_2) \\ \text{B} & (c_1, c_2) & (d_1, d_2) \end{array} \tag{6.1}$$

where a_i, b_i, c_i, $d_i > 0$, $i = 1, 2$. If the row player chooses the strategy A and the column player chooses B, then they get payoffs b_1 and b_2 respectively. Let us denote by x_i the fraction of population i that plays strategy A, $i = 1, 2$. (α_i, β_i) describes the personality profiles of the individuals in population i.

We define u^i_j as the attractiveness of the strategy $j = \{A, B\}$ in population $i \in \{1, 2\}$

$$\begin{aligned} u^i_A &= x_i^{1-\alpha_i} \nu_{A_i}^{1-\beta_i} \\ u^i_B &= (1 - x_i)^{1-\alpha_i} \nu_{B_i}^{1-\beta_i}. \end{aligned}$$

where ν_{ji} is the mean payoff of strategy j in population i.

Payoffs in population 1 equal

$$\begin{aligned} \nu_{A1} &= a_1 x_2 + b_1(1 - x_2) \\ \nu_{B1} &= c_1 x_2 + d_1(1 - x_2) \end{aligned}$$

and payoffs in population 2 are as follows

$$\begin{aligned} \nu_{A2} &= a_2 x_1 + c_2(1 - x_1) \\ \nu_{B2} &= b_2 x_1 + d_2(1 - x_1). \end{aligned}$$

The approach we use is similar to the one presented in previous chapters. However, let us note that in the case of asymmetric games, the social factor (popularity) depends on the composition of the considered population, whereas the economic factor (payoff) depends on the composition of the other population. Hence, members of each population determine their strategy observing the popularity of the available strategies in their own population and payoffs from the interactions with members of the other population.

The evolution equations take the form

$$\begin{cases} \dot{x_1} = u(\frac{u_A^1}{u} - x_1) = u_A^1 - x_1(u_A^1 + u_B^1) = (1-x_1)u_A^1 - x_1 u_B^1 \\ \dot{x_2} = (1-x_2)u_A^2 - x_2 u_B^2 \end{cases} \tag{6.2}$$

where $u = u_A^1 + u_B^2$. After substituting u_A^1, u_B^1, u_A^2 and u_B^2 values into the system (6.2), the dynamics for asymmetric games is as follows

$$\begin{cases} \dot{x_1} = (1-x_1)x_1^{1-\alpha_1}\nu_{A_1}^{1-\beta_1} - x_1(1-x_1)^{1-\alpha_1}\nu_{B1}^{1-\beta_1} \\ \dot{x_2} = (1-x_2)x_2^{1-\alpha_2}\nu_{A_2}^{1-\beta_2} - x_2(1-x_2)^{1-\alpha_2}\nu_{B2}^{1-\beta_2} \end{cases} \tag{6.3}$$

It is worth observing that for $\alpha_i = \beta_i = 0$, $i = 1,2$, we can obtain the standard replicator equation for asymmetric games

$$\begin{cases} \dot{x_1} = (1-x_1)x_1\nu_{A_1} - x_1(1-x_1)\nu_{B1} \\ \dot{x_2} = (1-x_2)x_2\nu_{A_2} - x_2(1-x_2)\nu_{B2} \end{cases}$$

Let us find the critical points of the system (6.3). The pure equilibria i.e. pairs of (x_1, x_2) such that x_1, $x_2 \in \{0,1\}$ are easy to determine: $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. In order to find mixed equilibria, we can substitute $z_i = \frac{x_i}{1-x_i}$, $i = 1,2$. Then the system (6.3) takes the form

$$\begin{cases} \dot{z}_1 = z_1^{1-\alpha_1}(1+z_1)_1^{\alpha}\nu_{A_1}^{1-\beta_1} - z_1(1+z_1)_1^{\alpha}\nu_{B_1}^{1-\beta_1} \\ \dot{z}_2 = z_2^{1-\alpha_2}(1+z_2)_1^{\alpha}\nu_{A_2}^{1-\beta_1} - z_2(1+z_2)_2^{\alpha}\nu_{B_2}^{1-\beta_2} \end{cases} \tag{6.4}$$

We obtain that the critical points of (6.4) $z^* = (z_1^*, z_2^*)$ satisfy the following conditions:

$$z_1^* = (\frac{a_1 z_2^* + b_1}{c_1 z_2^* + d_1})^{s_1} \text{ and } z_2^* = (\frac{a_2 z_1^* + b_2}{c_2 z_1^* + d_2})^{s_2}$$

where $s_i = \frac{1-\beta_i}{\alpha_i}$ is the sensitivity parameter for players in population i, $i = 1,2$.

Theorem 8. *System (6.3) does not have periodic solutions.*

Proof. We can consider the equation (6.4) since the stability properties of solutions of (6.4) are the same as solutions of (6.3).

We use Dulac's Criterion, which is recalled in Appendix A.3. Let us define a function

$$\phi(z_1, z_2) = [z_1(1+z_1)^{\alpha_1} z_2(1+x_2)^{\alpha_2}]^{-1}.$$

Applying Dulac's Criterion we get

$$\frac{\partial(\phi \dot{z}_1)}{\partial z_1} + \frac{\partial(\phi \dot{z}_2)}{\partial z_2} = -\frac{\alpha_1 z_1^{-(\alpha_1+1)}\nu_{A_1}^{1-\beta_1}}{z_2(1+z_2)^{\alpha_2}} - \frac{\alpha_2 z_2^{-(\alpha_2+1)}\nu_{A_2}^{1-\beta_2}}{z_1(1+z_1)^{\alpha_1}} < 0$$

for z_1, $z_2 > 0$. □

6.1.1. Coordination and anti-coordination games

Let us consider a pure asymmetric coordination game with $b_i = c_i = 0$, $a_i \neq 0$, $d_i \neq 0$, $i = 1, 2$. For $a_1 > d_1$ and $a_2 < d_2$, it corresponds to the well-known Battle of the Sexes game (see Appendix A.1). The payoff matrix is as follows

$$\begin{array}{c|cc} & \text{A} & \text{B} \\ \hline \text{A} & (a_1, a_2) & (0,0) \\ \text{B} & (0,0) & (d_1, d_2) \end{array} \tag{6.5}$$

The below theorem was proved in [7].

Theorem 9. *If $s_1 s_2 \neq 1$, then there exists a mixed equilibrium $z^* = (z_1^*, z_2^*)$ for the dynamics (6.4) and payoff matrix (6.5):*

$$z_1^* = (\frac{a_1 a_2^{s_2}}{d_1 d_2^{s_2}})^{\frac{s_1}{1-s_1 s_2}} \qquad and \qquad z_2^* = (\frac{a_1^{s_1} a_2}{d_1^{s_1} d_2})^{\frac{s_2}{1-s_1 s_2}}.$$

The above mixed equilibrium is locally asymptotically stable if $s_1 s_2 < 1$ and unstable, if $s_1 s_2 > 1$. If $s_1 s_2 = 1$, then for

- $(\frac{a_1}{d_1})^{s_1} \frac{a_2}{d_2}^{s_2} \neq 1$ *the mixed equilibrium does not exist*
- $(\frac{a_1}{d_1})^{s_1} \frac{a_2}{d_2}^{s_2} = 1$ *there exists a family of mixed equilibria $z_1^* = c$, $z_2^* = (c\frac{a_2}{d_2})^{s_2}$, where c is a positive constant.*

Similar theorem was obtained in [7] for an asymmetric anti-coordination game with the following payoff matrix

$$\begin{array}{c|cc} & \text{A} & \text{B} \\ \hline \text{A} & (0,0) & (b_1, b_2) \\ \text{B} & (c_1, c_2) & (0,0) \end{array} \tag{6.6}$$

Theorem 10. *If $s_1 s_2 \neq 1$, then there exists a mixed equilibrium $z^* = (z_1^*, z_2^*)$ for the dynamics (6.4) and payoff matrix (6.6):*

$$z_1^* = (\frac{b_1 c_2^{s_2}}{c_1 b_2^{s_2}})^{\frac{s_1}{1-s_1 s_2}} \qquad and \qquad z_2^* = (\frac{c_1^{s_1} b_2}{b_1^{s_1} c_2})^{\frac{s_2}{1-s_1 s_2}}.$$

The above mixed equilibrium is locally asymptotically stable if $s_1 s_2 < 1$ and unstable, if $s_1 s_2 > 1$. If $s_1 s_2 = 1$, then for

- $(\frac{b_1}{c_1})^{s_1} \frac{c_2}{b_2} \neq 1$ *the mixed equilibrium does not exist*
- $(\frac{b_1}{c_1})^{s_1} \frac{c_2}{b_2} = 1$ *there exists a family of mixed equilibria $z_1^* = c$, $z_2^* = (\frac{b_2}{c_2})^{s_2}$, where c is a positive constant.*

6.2. Nonconformist preferences

In this section, we focus on a two-person asymmetric game with two nonconformist preferences. In such a case, the attractiveness functions for strategies A and B respectively equal

$$u_A^i = (1 - x_i)^{1-\alpha_i} \nu_{A_i}^{1-\beta_i}$$
$$u_B^i = x_i^{1-\alpha_i} \nu_{A_i}^{1-\beta_i}$$

where $i = 1, 2$. The evolution equations are as follows

$$\begin{cases} \dot{x_1} = (1 - x_1)(1 - x_1)^{1-\alpha_1}\nu_{A_1}^{1-\beta_1} - x_1 x_1^{1-\alpha_1}\nu_{B_1}^{1-\beta_1} = (1 - x_1)^{2-\alpha_1}\nu_{A_1}^{1-\beta_1} - x_1^{2-\alpha_2}\nu_{B_1}^{1-\beta_1} \\ \dot{x_2} = (1 - x_2)^{2-\alpha_2}\nu_{A_2}^{1-\beta_2} - x_2^{2-\alpha_2}\nu_{B_2}^{1-\beta_2} \end{cases} \tag{6.7}$$

After substituting $z_i = \frac{x_i}{1-x_i}$ to (6.7), we get the following system

$$\begin{cases} \dot{z}_1 = (1 + z_1)^{\alpha_1}\nu_{A_1}^{1-\beta_1} - z_1^{2-\alpha_1}(1 + z_1)^{\alpha_1}\nu_{B_1}^{1-\beta_1} \\ \dot{z}_2 = (1 + z_2)^{\alpha_2}\nu_{A_2}^{1-\beta_2} - z_2^{2-\alpha_2}(1 + z_2)^{\alpha_2}\nu_{B_2}^{1-\beta_2} \end{cases} \tag{6.8}$$

We want to find the equilibrium $z^* = (z_1^*, z_2^*)$ of the system (6.8) i.e. a point for which the right hand sides of these equations are equal to zero. After straightforward transformations, we obtain

$$z_i^* = (\frac{v_{A_i}}{v_{B_i}})^{\frac{1-\beta_i}{2-\alpha_i}}$$

where $i = 1, 2$.

Let us consider now the payoff matrix of the coordination game (6.5). Then, payoffs from choosing strategy A or B by players from population 1 are equal to

$$\nu_{A_1} = \frac{a_1 z_2}{1 + z_2} \qquad \text{and} \qquad \nu_{B_1} = \frac{d_1}{1 + z_2}.$$

The evolution equations take the form

$$\begin{cases} \dot{z}_1 = (1 + z_1)^{\alpha_1}(\frac{a_1 z_2}{1 + z_2})^{1-\beta_1} - z_1^{2-\alpha_1}(1 + z_1)^{\alpha_1}(\frac{d_1}{1 + z_2})^{1-\beta_1} \\ \dot{z}_2 = (1 + z_2)^{\alpha_2}(\frac{a_2 z_1}{1 + z_1})^{1-\beta_2} - z_2^{2-\alpha_2}(1 + z_2)^{\alpha_2}(\frac{d_2}{1 + z_1})^{1-\beta_2} \end{cases} \tag{6.9}$$

The critical point of the system (6.9) can be easily computed

$$z^* = ((\frac{a_1}{d_1}(\frac{a_2}{d_2})^{\tilde{s}_2})^{\frac{\tilde{s}_1}{1-\tilde{s}_1\tilde{s}_2}}, (\frac{a_2}{d_2}(\frac{a_1}{d_1})^{\tilde{s}_1})^{\frac{\tilde{s}_2}{1-\tilde{s}_1\tilde{s}_2}})$$

where $\tilde{s_i} = \frac{1-\beta_i}{2-\alpha_i}$, $i = 1, 2$.
Now, we linearize the system (6.9). Let us denote by

$$f_1(z_1, z_2) = (1 + z_1)^{\alpha_1}(\frac{a_1 z_2}{1 + z_2})^{1-\beta_1} - z_1^{2-\alpha_1}(1 + z_1)^{\alpha_1}(\frac{d_1}{1 + z_2})^{1-\beta_1}$$
$$f_2(z_1, z_2) = (1 + z_2)^{\alpha_2}(\frac{a_2 z_1}{1 + z_1})^{1-\beta_2} - z_2^{2-\alpha_2}(1 + z_2)^{\alpha_2}(\frac{d_2}{1 + z_1})^{1-\beta_2}$$

and calculate the partial derivatives of $f_1(z_1, z_2)$

$$\frac{\partial f_1}{\partial z_1} = -(\frac{d_1}{1 + z_2})^{1-\beta_1}(1 + z_1)^{\alpha_1} z_1 1 - \alpha_1(2 - \alpha_1)$$
$$\frac{\partial f_1}{\partial z_2} = (1 + z_1)^{\alpha_1} z_2^{-1}(1 - \beta_1)(\frac{a_1 z_2}{1 + z_2})^{1-\beta_1}.$$

We can calculate the partial derivatives of $f_2(z_1, z_2)$ similarly. The linearization matrix of the system (6.9) at the point z^* has the form

$$\begin{bmatrix} -(\frac{d_1}{1+z_2^*})^{1-\beta_1}(1 + z_1^*)^{\alpha_1}(z_1^*)^{1-\alpha_1}(2 - \alpha_1) & (1 + z_1^*)^{\alpha_1}(z_2^*)^{-1}(1 - \beta_1)(\frac{a_1 z_2^*}{1+z_2^*})^{1-\beta_1} \\ (1 + z_2^*)^{\alpha_2}(z_1^*)^{-1}(1 - \beta_2)(\frac{a_2 z_1^*}{1+z_1^*})^{1-\beta_2} & -(\frac{d_2}{1+z_1^*})^{1-\beta_2}(1 + z_2^*)^{\alpha_2}(z_2^*)^{1-\alpha_2}(2 - \alpha_2) \end{bmatrix}$$

Both eigenvalues of the linearization matrix have negative real parts if and only if the determinant of the linearization matrix is positive and the trace of this matrix is negative (see Appendix A.4). In our case, it is clear to see that trace of the matrix is always negative. Determinant of the matrix is positive, when

$$\tilde{s}_1\tilde{s}_2 < (z_1^*)^{2(1-\alpha_1)}(z_2^*)^{2(1-\alpha_2)}. \tag{6.10}$$

After substituting z_1^* and z_2^* values into the inequality (6.10), we can state that the mixed equilibrium is locally asymptotically stable if and only if the parameters $\tilde{s}_1\tilde{s}_2 \neq 1$ and $\tilde{s}_1$, $\tilde{s}_2$, α_1, α_2 satisfy the following condition

$$\tilde{s}_1\tilde{s}_2 < [\frac{a_1}{d_1}(\frac{a_2}{d_2})^{\tilde{s}_2}]^{\frac{2\tilde{s}_1(1-\alpha_1)}{1-\tilde{s}_1\tilde{s}_2}}[\frac{a_2}{d_2}(\frac{a_1}{d_1})^{\tilde{s}_1}]^{\frac{2\tilde{s}_2(1-\alpha_2)}{1-\tilde{s}_1\tilde{s}_2}}.$$

We can also prove that in the case of asymmetric games, there are no periodic solutions. We will use the Dulac's Criterion, similarly as in Section 6.1.

Theorem 11. *System (6.7) does not have periodic solutions.*

Proof. Similarly as earlier, we can consider the equation (6.8) since the solutions to this equation have the same stability properties as the solutions of (6.7).

Let us define a function $\phi(z_1, z_2) = [(1+z_1)^{\alpha_1}(1+z_2)^{\alpha_2}]^{-1}$ and calculate relevant partial derivatives:

$$\frac{\partial(\phi\dot{z}_1)}{\partial z_1} = -(2-\alpha_1)z_1^{1-\alpha_1}\nu_{B_1}^{1-\beta_1}(1+z_2)^{-\alpha_2}$$
$$\frac{\partial(\phi\dot{z}_2)}{\partial z_2} = -(2-\alpha_2)z_2^{1-\alpha_2}\nu_{B_1}^{1-\beta_2}(1+z_1)^{-\alpha_1}$$

Applying Dulac's criterion we get that

$$\frac{\partial(\phi\dot{z}_1)}{\partial z_1} + \frac{\partial(\phi\dot{z}_2)}{\partial z_2} < 0 \Leftrightarrow \alpha_1 < 2 \text{ and } \alpha_2 < 2.$$

These conditions are always satisfied, as we assume that $\alpha_i \in [0, 1]$, $i = 1, 2$. □

Conclusions

The aim of my thesis was to study evolutionary dynamics of populations of individuals with complex personality profiles. The dynamics was described using the idea of the attractiveness function that reflects the broader spectrum of social interactions, compared to the standard replicator dynamics. All of the results were obtained for infinite populations with random pairwise matching.

After brief characterization of the general idea behind the proposed model and introducing attractiveness function, two-person symmetric games with two strategies were analyzed (Prisoner's Dilemma, coordination and anti-coordination games). Then the two-person symmetric games with three strategies were considered (mainly the Rock-Paper-Scissors game). In next chapters, we expanded the model by modifying the attractiveness function. We incorporated to this function not only popularity and payoffs, but also other factors, namely a constant, transcendent factor, rate factor that takes into account rate of change in popularity and selection potential, which includes the variance of payoffs. Such a general attractiveness function may better reflect the mechanisms that govern the decision making of individuals. The concept of nonconformist preferences was also introduced and some analytical results were obtained.

The important part of this work concerned delays introduced to the attractiveness function. We checked how the delays may influence the dynamics in various games with two strategies (Prisoner's Dilemma game, coordination and anti-coordination games), as well as in games with three strategies (Rock-Paper-Scissors game, Weak Iterated Prisoner's Dilemma game). As expected, it appeared that delays may have a strong impact on the dynamics of our model. It is worth stressing that the chapter regarding delays is preliminary and exploratory.

Further, we introduced the two-person asymmetric games with two strategies and proved using the Dulac's criterion that in such a situation there are no periodic solutions under given dynamics. We also formulated theorems concerning particular types of games, namely the coordination and anti-coordination games.

The presented model can serve to explain some theoretical and sociological problems. Apart from structural simplicity of games, they may cover complex individual behaviour, motivation of agents and rules of searching for stable solutions. Obviously, there is a need for further investigation of the model. Many interesting problems connected, for instance, with a special dynamics for some values of parameters, have appeared in this work. Based on numerical calculations, we can observe strange dynamics that may be connected with existence of a secondary fold bifurcation. The second important problem concerns introducing delays to the attractiveness function and their influence on the dynamics. There is also a possibility to expand the concept of asymmetric games, consider changing personalities of the actors during the interactions or models with finite number of heterogeneous agents with different personality profiles. As it is mentioned above, there are various open problems related to the presented research and they may become an interesting topic of the future work in this field.

Appendix

A.1 Brief descriptions of games with two strategies

In this section, the brief descriptions of games that have been mentioned earlier are presented. In the symmetric games described below, individuals choose from two strategies C and D, and the payoff matrix $[R, S, T, P]$ has the following form

	C	D
C	R	S
D	T	P

where $R, S, T, P \geq 0$.

1. **Prisoner's Dilemma**
   A Prisoner's Dilemma game is one of the most often analyzed games. In this game, payoffs have to satisfy the following conditions

   $$T > R > P > S.$$

   If $T > R > P = S$, then such a game is called Weak Prisoner's Dilemma game.

   Pair (D, D) is the only pure Nash equilibrium i.e. such a set of strategy choices, when each player has chosen a strategy and no player can benefit by changing his or her strategy unilaterally. The Prisoner's Dilemma game does not have any mixed equilibria under the standard replicator dynamics i.e. there are no stable solutions with nonzero frequency of cooperators and defenders.

   The name of this game derives from a story about two prisoners.

   Two persons are suspected of committing a crime and are being interrogated in two separate rooms. They cannot communicate. Both of them want to minimize their jail sentence. Investigators do not have enough proofs to sentence both of them and offer both a similar deal – if one testifies against his partner i.e. defects and chooses D and the other remains silent i.e. cooperates and chooses C, the betrayer goes free and the cooperator receives the five-year sentence. If both remain silent, both are sentenced to only one year in jail for a minor charge. If each betray the other, each receives a three year sentence. Each prisoner must choose either to betray or remain silent.

   In this situation, the payoff matrix is as follows

	C	D
C	3	0
D	5	1

2. **Coordination game**

In a coordination game payoffs satisfy the following conditions

$$R > T \text{ and } P > S.$$

Coordination games are a class of games with two pure Nash equilibria. This type of games illustrate a problem of coordination i.e. situations in which both agents can realize mutual aims, but only by making consistent decisions.

Pairs (C, C) and (D, D) are pure Nash equilibria. Coordination games also have mixed equilibrium, which is unstable under the replicator dynamics.

A typical example of the coordination game is a game, when players choose the side of the road upon which to drive.

Assume that two drivers meet on a narrow road. Both have to swerve in order to avoid a head-on collision. If both of them execute the same swerving maneuver, they will manage to avoid the collision, but if they choose different maneuvers they will collide.

In the below payoff matrix, successful choice is represented by a payoff of 10, and a collision by a payoff of 0

	C	D
C	10	0
D	0	10

- **Stag Hunt**

  A Stag Hunt game is an example of the coordination game, but with the additional condition $T \geq P$. The name of this games derives from the story about hunting.

  Two agents decide to hunt for hares or stag, which has the greater value for them than hares. Their decisions are simultaneous and independent. The only way to successfully hunt the stag is to work together, so both of them should choose C. If one of the hunters focuses on the stag and the other focuses on a hare, so one of them chooses C and the other D, then the first one will end up with nothing and the second one will end up with two hares. If both individuals hunt for hares, so both chooses D, then each will kill one hare.

  The payoff matrix is as follows

	C	D
C	2	0
D	1	1

- **Battle of the Sexes**

  A Battle of the Sexes game is a two-player asymmetric coordination game that describes the following scenario:

  A couple is meeting this evening, but cannot recall whether they decided to attend a ballet (strategy C) or a football match (strategy D). The wife would prefer to see the ballet. The husband would like to see the football match. Both of them would prefer to go to the same place. The game is asymmetric since the payoff obtained for attending ballet is smaller for husband than for wife and analogously for a football match.

An example payoff matrix is given below. The wife is a row player and the husband chooses a column. In each cell, the first number represents the payoff to the wife and the second number represents the payoff to the husband.

	C	D
C	$3,2$	$0,0$
D	$0,0$	$2,3$

3. **Anti-coordination game**

   In an anti-coordination game, payoffs satisfy the following conditions

   $$T > R \text{ and } S > P.$$

   The anti-coordination problem is opposite to the previous one. Here, both players benefit from choosing opposite decisions.

   The game has two pure Nash equilibria i.e. (C, D) and (D, C) and one mixed equilibrium, which is stable under replicator dynamics.

   - **Snow Drift**

     To get a Snow-Drift game, payoffs have to satisfy the following conditions

     $$T > R > S > P.$$

     The name of this game derives from the following story:

     Two drivers are trapped on two different sides of a snowdrift. Cooperation means to get out of the car and shovel. Removing the snowdrift costs c. If both drivers cooperate, then the cost of shoveling is c/2. Defection means to remain in the car and wait till the other one do the work. If at least one of the drivers cooperates, then both gain the benefit b of getting home. It is assumed that b > c.

     The payoff matrix of this game for $b = 4$, $c = 2$ is

	C	D
C	3	2
D	4	0

A.2 Delay differential equations

The classical model based on the ordinary differential equations assumes that in a given moment of time the reaction of system is immediate and depends on the state of the system in this moment i.e. $y'(t) = f(t, y(t))$. However, there are situations, when such an assumption is unrealistic. Thus, it is reasonable to take into account the fact that the derivative of the solution may depend not only on the current situation, but also on the state of the system in earlier moments.

In this work, we focus on the autonomous delay differential equations with the initial condition that is a constant function. The full definition of delay differential equations is very expanded and formal, and it is not the subject of this thesis. Therefore, only the brief description is provided below. Interested readers can find the complete information about delay differential equations with precise explanation in [4].

Let X be an arbitrary space and φ an arbitrary function defined on some segment I that takes values in X. Moreover let τ be a finite real number and $\tau \in I$.

Definition. *For each $t \in I \subset \mathbb{R}$ we define a function $\varphi_t : [-\tau, 0] \to X$ in the following way*

$$\varphi_t(s) = \varphi(s+t) \text{ for each } s \in [-\tau, 0].$$

Definition. *The following system is called the system of differential equations with delay*

$$\dot{x}(t) = F(t, x(t-\tau)) \qquad \text{for } t \geq t_0, \tag{7.1}$$

where

- $x(t) \in \mathbb{R}^n$
- $F : [b, +\infty) \times \mathcal{C} \to \mathbb{R}^n$ *is a given function, b is an arbitrary real number, $t_0 \geq b$ and $\mathcal{C}$ is a Banach space of continuous functions defined on interval $[-\tau, 0]$ and taking values in $\mathbb{R}^n$ with the standard supremum norm i.e. $\mathcal{C} = C([-\tau, 0]; \mathbb{R}^n)$.*

The initial condition for the system (7.1) is

$$x(t) = \varphi(t - t_0) \qquad \text{for } t \in [t_0 - \tau, t_0].$$

where $\varphi \in \mathcal{C}$.

In the most common situations, increasing the delay results in destabilization. However, it should be remembered that sometimes the situation may be different. Although, the stable state usually loses its stability with increasing delay, it may also happen that the unstable gains the stability. The oscillations and periodic solutions may also appear. In [4], it was proved that for one equation with only one discrete delay τ, the increase of τ can only destabilize the stationary state.

In general, different scenarios of the influence of a discrete delay on the stationary state are possible:

- the delay has no effect on the stability of the stationary state
- sufficiently large delay destabilizes the stationary state which is stable without the delay
- the appropriate values of delay stabilizes the stationary state which is unstable without the delay.

There are theorems concerning existence and uniqueness of solutions to the delay differential equations. They are similar to theorems formulated for ordinary differential equations. Let us consider

$$\dot{x}(t) = F(t, x_t) \qquad \text{for } t \geq t_0 \tag{7.2}$$

$$x(t) = \varphi(t - t_0) \qquad \text{for } t \in [t_0 - \tau, t_0],\ \varphi \in \mathcal{C} \tag{7.3}$$

Theorem 12. ***(Existence of solutions)*** *Let Ω be an open subset $\mathbb{R}^n \times \mathcal{C}$ and let the function F be a continuous function defined on Ω. Then for each point $(t_0, \varphi) \in \Omega$, there exists a local solution of equation (7.2) with the given initial condition.*

Theorem 13. ***(Uniqueness of solutions)*** *Let Ω be an open subset $\mathbb{R}^n \times \mathcal{C}$ and let F be a continuous function defined on Ω and satisfying the Lipschitz condition with respect to the second variable on each compact subset of Ω. Then for each point $(t_0, \varphi) \in \Omega$, the solution of equation (7.2) with the given initial condition is unique.*

A.3 Criteria for excluding periodic orbits

A periodic orbit corresponds to the solution of a dynamical system, which repeats itself in time. It is sometimes possible to prove analytically that a periodic orbit does not exist using the Bendixson's and Dulac's criteria. They apply for an autonomous planar vector field

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

where $(x, y) \in R^2$.

Theorem 14. *(**Bendixon's criterion**) Let D be a simply connected region $D \subset \mathbb{R}^2$. If*

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$$

is not identically zero and does not change sign in D, then there are no periodic orbits lying entirely in D.

Theorem 15. *(**Dulac's criterion**) Let $B(x, y)$ be a scalar function defined on a simply connected region $D \subset \mathbb{R}^2$. If*

$$\frac{\partial(BF)}{\partial x} + \frac{\partial(BG)}{\partial y}$$

is not identically zero and does not change sign in D, then there are no periodic orbits lying entirely in D.

Dulac's criterion is a generalization of Bendixson's criterion, which corresponds to the function $B(x, y) \equiv 1$.

A.4 Characteristic polynomial

Let us focus on a linear system of two differential equations with constant coefficients. We consider the homogeneous linear equation

$$\dot{x} = Ax \tag{7.4}$$

with a constant matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The critical point of the equation (7.4) is $x^* = 0$. Assume that $detA \neq 0$. Let us consider the characteristic polynomial of the matrix A

$$det(A - \lambda I) = det(\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} =$$
$$= \lambda^2 - (a_{11} + a_{21})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

where I is the identity matrix. Hence, the characteristic polynomial $w(\lambda)$ can be written in the form

$$w(\lambda) = \lambda^2 - (trA)\lambda + detA.$$

The characteristic values of the matrix A equal:

$$\lambda_1 = \frac{1}{2}(trA + \sqrt{\Delta}) \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(trA - \sqrt{\Delta})$$

where $\Delta = (trA)^2 - 4detA$.

To have the stability of the critical point x^*, real parts of both eigenvalues have to be smaller than 0. Using Vieta's formulas, it can be easily checked, when the real parts of the eigenvalues are negative. The sum and the multiplication of characteristic values λ_1, λ_2 have to satisfy the following conditions

$$\lambda_1 + \lambda_2 = \frac{-b}{a} = trA$$
$$\lambda_1 \lambda_2 = \frac{c}{a} = detA$$

Therefore, in order to have both real parts of eigenvalues negative, the trace of matrix A has to be negative and the determinant of matrix A has to be positive.

Bibliography

[1] T. Płatkowski, J. Poleszczuk, *Operant Response Theory of Social Interactions*, Journal of Biological Sciences, Volume 1, Issue 1, 2009, p. 1-10.

[2] T. Płatkowski, *Cooperation in two-person evolutionary games with complex personality profiles*, Journal of Theoretical Biology, Volume 266, 2010, p. 522-528.

[3] T. Płatkowski, J. Zakrzewski, *Asymptotically stable equilibrium and limit cycles in the Rock-Paper-Scissors game in a population of players with complex personalities*, Physica A, Volume 390, Issue 23-24, 2011, p. 4219-4226.

[4] M. Bodnar, M. J. Piotrowska, *O równaniach różniczkowych z opóźnieniem – teoria i zastosowania*, Matematyka Stosowana, Volume 11/52, 2012.

[5] M. A. Nowak, R. M., *Evolutionary games and spatial chaos*, Nature, Volume 359, 1992, p. 826-829.

[6] J. Poleszczuk, *Ewolucyjna teoria interakcji społecznych*, Wydawnictwo Naukowe Scholar, Warszawa, 2004.

[7] T. Płatkowski, J. Zakrzewski, *Game Dynamics for Players with Social and Material Preferences*, arXiv:1203.6749v1 [nlin.AO].

[8] A. Palczewski, *Równania różniczkowe zwyczajne*, Wydawnictwa Naukowo-Techniczne, Warszawa 2004.

[9] W. H. Sandholm, E. Dokumaci, and F. Franchetti (2012), Dynamo: Diagrams for Evolutionary Game Dynamics, `http://www.ssc.wisc.edu/~whs/dynamo`.

[10] Jeff Moehlis et al. (2006) Periodic orbit, Scholarpedia, 1(7): 1358, `http://www.scholarpedia.org/article/Periodic_orbit`.